

M337

Complex analysis

Book B

Integration of complex functions

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Unit B1

Integration



# Introduction

In this unit we introduce *complex integration*, an important concept which gives complex analysis its special flavour. We spend most of this unit setting up the complex integral, deriving its main properties, and illustrating various techniques for evaluating it. In later units of the module we discuss the uses of complex integrals in complex analysis.

To define the integral of a complex function, it is instructive to first consider real integrals, such as

$$\int_a^b x^2 dx = \frac{1}{3}(b^3 - a^3),$$

where  $a < b$ , which represents the area of the shaded part of Figure 0.1 (for  $a > 0$ ). We can express this equation in words by saying that

the integral of the function  $f(x) = x^2$  over the interval  $[a, b]$  is  $\frac{1}{3}(b^3 - a^3)$ .

Suppose now that we wish to integrate the complex function  $f(z) = z^2$  between two points  $\alpha$  and  $\beta$  in the complex plane. To do this, we first need to specify exactly how to get from  $\alpha$  to  $\beta$ . We could, for example, choose the line segment  $\Gamma$  from  $\alpha$  to  $\beta$ , as shown in Figure 0.2. It turns out (as you will see later) that if we make this choice, then

the integral of the function  $f(z) = z^2$  along the line segment from  $\alpha$  to  $\beta$  is  $\frac{1}{3}(\beta^3 - \alpha^3)$ .

We write this as

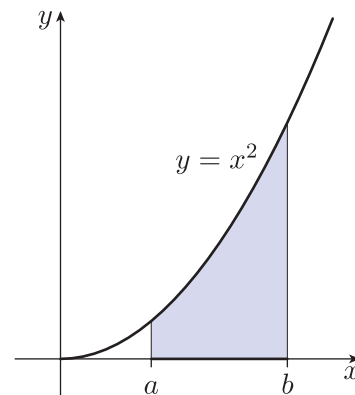
$$\int_{\Gamma} z^2 dz = \frac{1}{3}(\beta^3 - \alpha^3).$$

But there are many other paths in the complex plane from  $\alpha$  to  $\beta$ , which raises the following question. Do we get the same answer if we integrate the function  $f(z) = z^2$  along a different path from  $\alpha$  to  $\beta$ ?

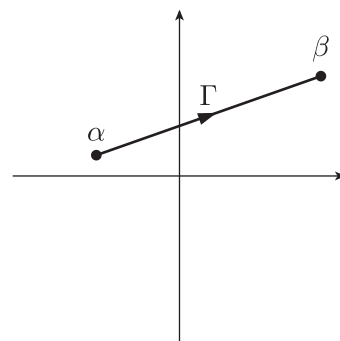
In order to address this question, we first need to explain exactly what it means to ‘integrate a function along a path’. This is one of the objectives of Section 1, where we briefly review the Riemann integral from real analysis, and then use similar ideas to construct the integral of a complex function along a path in the complex plane. We will see that if  $f$  is a complex function that is continuous on a smooth path  $\Gamma : \gamma(t)$  ( $t \in [a, b]$ ) in the complex plane, then the integral of  $f$  along  $\Gamma$ , denoted by

$\int_{\Gamma} f(z) dz$ , is given by the formula

$$\int_{\Gamma} f(z) dz = \int_a^b f(\gamma(t)) \gamma'(t) dt.$$



**Figure 0.1** Area under the graph of  $y = x^2$  between  $a$  and  $b$



**Figure 0.2** Line segment  $\Gamma$  from  $\alpha$  to  $\beta$

We can evaluate this integral by splitting  $f(\gamma(t))\gamma'(t)$  into its real and imaginary parts  $u(t)$  and  $v(t)$ , and evaluating the resulting pair of *real* integrals:

$$\int_a^b f(\gamma(t))\gamma'(t) dt = \int_a^b u(t) dt + i \int_a^b v(t) dt.$$

Section 2 begins with this definition of the integral of a complex function along a smooth path, and then extends the idea to allow integration along a *contour* – a finite sequence of smooth paths laid end to end.

In Section 3 we prove the Fundamental Theorem of Calculus, which shows that integration and differentiation are essentially inverse processes. From this result it follows that the integral of  $f(z) = z^2$  along *any* contour from  $\alpha$  to  $\beta$  is  $\frac{1}{3}(\beta^3 - \alpha^3)$ .

We will need to be careful about how we apply results such as the Fundamental Theorem of Calculus. For example, suppose that the endpoints  $\alpha$  and  $\beta$  of  $\Gamma$  coincide, as illustrated in Figure 0.3. Then

$$\int_{\Gamma} z^2 dz = \frac{1}{3}(\beta^3 - \alpha^3) = 0.$$

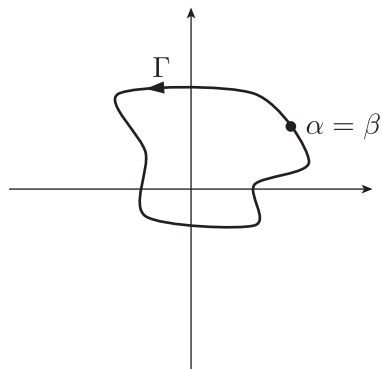
In this case, the integral of the function  $f(z) = z^2$  along  $\Gamma$  is 0.

Consider now the function  $f(z) = 1/z$ . We will see in Example 2.3 and Exercise 3.5(a) that if we integrate  $f$  along the smooth paths  $\Gamma_1$  and  $\Gamma_2$  shown in Figure 0.4, where  $\Gamma_1$  and  $\Gamma_2$  are circles traversed once anticlockwise, then

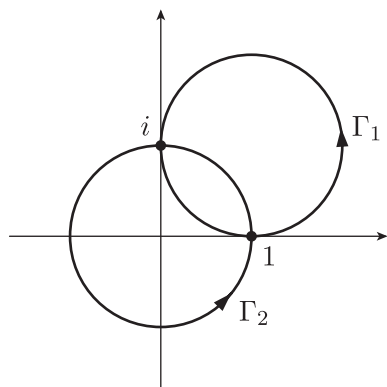
$$\int_{\Gamma_1} \frac{1}{z} dz = 0, \quad \text{but} \quad \int_{\Gamma_2} \frac{1}{z} dz = 2\pi i.$$

The reason for this difference will become apparent in Section 3, and indeed we return to examples of this type many times throughout Book B.

Later in the module we will meet complex integrals whose values we cannot determine exactly. We can, however, *estimate* them. In Section 4 we present the Estimation Theorem, which is a useful technique for obtaining an upper estimate for the modulus of an integral. This theorem is much used in complex analysis.



**Figure 0.3** Contour with initial point  $\alpha$  and final point  $\beta$  coinciding



**Figure 0.4** Circular paths  $\Gamma_1$  and  $\Gamma_2$

## Unit guide

You may wish to skim through Section 1 if you are familiar with real integration, and then proceed to Section 2, which begins with the definition of the integral of a complex function. Make sure you are comfortable with the material of Section 2 as it is a fundamental part of the module.

You should also make sure that you understand the statement of the Fundamental Theorem of Calculus (in Subsection 3.1) and the Estimation Theorem (in Subsection 4.2), as both feature significantly in later units.

# 1 Integrating real functions

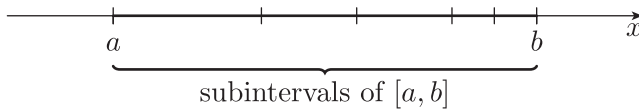
After working through this section, you should be able to:

- appreciate how the *Riemann integral* is defined
- state the main properties of the Riemann integral
- appreciate how complex integrals can be defined.

In this section we define the Riemann integral of a continuous real function (named after Bernhard Riemann, whom we met in Book A for the Cauchy–Riemann equations) and outline its main properties. We then discuss complex integrals.

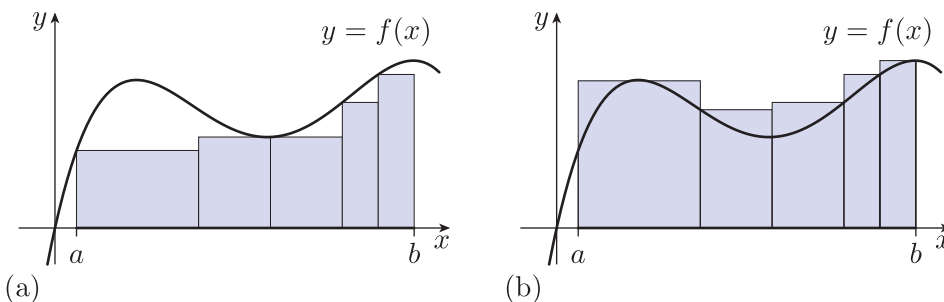
One of the uses of real integration is to determine the area under a curve. For example, the integral of a continuous function  $f$  that takes only positive values between the real numbers  $a$  and  $b$ , where  $a < b$ , is the area bounded by the graph of  $y = f(x)$ , the  $x$ -axis, and the two vertical lines  $x = a$  and  $x = b$ , as illustrated by the shaded part of Figure 1.1.

We can estimate this area by first splitting the interval  $[a, b]$  into a finite number of subintervals, such as those shown in Figure 1.2.



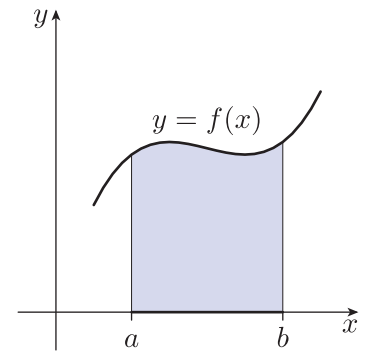
**Figure 1.2** Interval  $[a, b]$  split into subintervals

We can then underestimate the area under the graph of  $y = f(x)$  between  $a$  and  $b$  by summing the areas of those rectangles that have the various subintervals as bases and for which the top edge of each rectangle touches the graph from below, as shown in Figure 1.3(a). Similarly, we can overestimate the area under  $y = f(x)$  between  $a$  and  $b$  by summing the areas of those rectangles that have the various subintervals as bases and for which the top edge of each rectangle touches the graph from above, as shown in Figure 1.3(b).



**Figure 1.3** (a) An underestimate (b) An overestimate

We now let the number of subintervals tend to infinity, in such a way that the lengths of the subintervals tend to zero. It can be shown that the underestimates and overestimates of the area tend to a common limit  $A$ ,



**Figure 1.1** Area under the graph of  $y = f(x)$  between  $a$  and  $b$

written as

$$A = \int_a^b f(x) dx.$$

We call  $A$  the *area under the graph of  $y = f(x)$  between  $a$  and  $b$* .

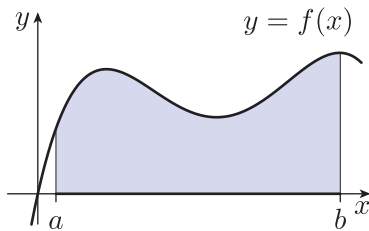
This underestimating and overestimating approach is often how Riemann integration is first introduced, and you may have seen it before. However, we encounter a problem if we try to generalise this particular approach to complex functions. Inequalities between complex numbers have no meaning, so it makes no sense to try to estimate complex numbers from ‘below’ or ‘above’. To get round this problem, we now outline a different approach to defining the integral of a real function – one that does generalise to complex functions.

Rather than underestimating and overestimating the area under the curve with rectangles, we choose a single point inside each subinterval and use this to construct a rectangle whose base is the subinterval, and whose height is the value of the function at the chosen point. The sum of the areas of these rectangles should then be an approximation to the area under the graph. As long as our function  $f$  is continuous on the interval  $[a, b]$ , then this modified approach (which does generalise to complex integrals) agrees with the underestimating and overestimating approach.

In this section we use this modified approach to give a formal definition of the Riemann integral, and then we summarise the main properties of the Riemann integral. We omit all proofs, which can be found in texts on real analysis, and which you may have seen in other modules.

## 1.1 Integration on the real line

We wish to define the Riemann integral of a continuous real function  $f$  in such a way that if  $f$  is positive on some interval  $[a, b]$ , then the integral of  $f$  from  $a$  to  $b$  is the area under the graph of  $y = f(x)$  between  $a$  and  $b$ . This is illustrated by the shaded part of Figure 1.4. To do this, we first split the interval  $[a, b]$  into a collection of subintervals called a *partition*.



**Figure 1.4** Area under the graph of  $y = f(x)$  between  $a$  and  $b$

### Definitions

A **partition**  $P$  of the interval  $[a, b]$  is a finite collection of subintervals of  $[a, b]$ ,

$$P = \{[x_0, x_1], [x_1, x_2], \dots, [x_{n-1}, x_n]\},$$

for which

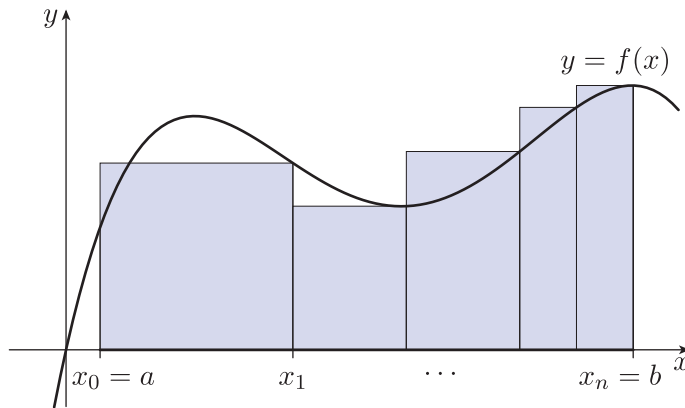
$$a = x_0 \leq x_1 \leq x_2 \leq \dots \leq x_n = b.$$

The **length** of the subinterval  $[x_{k-1}, x_k]$  is  $\delta x_k = x_k - x_{k-1}$ .

We use  $\|P\|$  to denote the maximum length of all the subintervals, so

$$\|P\| = \max\{\delta x_1, \delta x_2, \dots, \delta x_n\}.$$

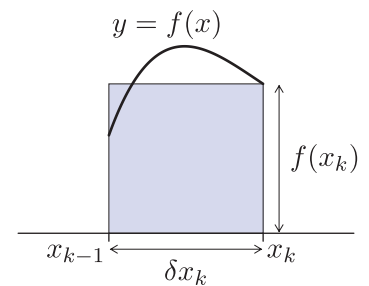
Given a partition  $P = \{[x_0, x_1], [x_1, x_2], \dots, [x_{n-1}, x_n]\}$  of  $[a, b]$ , we can approximate the area under the graph of  $y = f(x)$  between  $a$  and  $b$  by constructing a sequence of rectangles, as shown in Figure 1.5.



**Figure 1.5** Approximating the area under a graph using a sequence of rectangles

Here the  $k$ th rectangle has base  $[x_{k-1}, x_k]$  and height  $f(x_k)$  (so the top-right corner of the rectangle touches the curve). The area of the rectangle is  $f(x_k)(x_k - x_{k-1})$  (see Figure 1.6). Note that we could equally have chosen the rectangle to be of height  $f(c_k)$  for any point  $c_k$  in  $[x_{k-1}, x_k]$ , and the theory would still work. This is because, for a continuous function  $f$ , the difference between one set of choices of values for  $c_k$ ,  $k = 1, 2, \dots, n$ , and another disappears when we take limits of partitions. We have chosen  $f(x_k)$  merely for convenience.

Summing the areas of all the rectangles gives an approximation to the area under the graph. This sum is called the *Riemann sum* for  $f$ , with respect to this particular partition. (You may have seen *upper Riemann sum* and *lower Riemann sum* defined slightly differently elsewhere.)



**Figure 1.6** Rectangle of height  $f(x_k)$  and width  $\delta x_k$

### Definition

The **Riemann sum** for  $f$  with respect to the partition

$$P = \{[x_0, x_1], [x_1, x_2], \dots, [x_{n-1}, x_n]\}$$

is the sum

$$R(f, P) = \sum_{k=1}^n f(x_k) \delta x_k = \sum_{k=1}^n f(x_k)(x_k - x_{k-1}).$$

We now calculate the Riemann sum for a particular choice of function and partition, and then ask you to do the same for a second function.

**Example 1.1**

Let  $f(x) = x^2$ , where  $x \in [0, 1]$ . Show that for

$$P_n = \{[0, 1/n], [1/n, 2/n], \dots, [(n-1)/n, 1]\},$$

we have

$$R(f, P_n) = \frac{1}{6}(1 + 1/n)(2 + 1/n),$$

and determine  $\lim_{n \rightarrow \infty} R(f, P_n)$ .

**Solution**

Each of the  $n$  subintervals of  $P_n$  has length  $1/n$ . Therefore

$$\begin{aligned} R(f, P_n) &= \sum_{k=1}^n f\left(\frac{k}{n}\right) \times \frac{1}{n} \\ &= \sum_{k=1}^n \left(\frac{k}{n}\right)^2 \times \frac{1}{n} \\ &= \frac{1}{n^3} \sum_{k=1}^n k^2. \end{aligned}$$

Using the identity

$$\sum_{k=1}^n k^2 = 1^2 + 2^2 + \dots + n^2 = \frac{1}{6}n(n+1)(2n+1),$$

we obtain

$$R(f, P_n) = \frac{1}{n^3} \times \frac{1}{6}n(n+1)(2n+1) = \frac{1}{6}(1 + 1/n)(2 + 1/n),$$

as required.

Finally, since  $(1/n)$  is a basic null sequence, we see that

$$\lim_{n \rightarrow \infty} R(f, P_n) = \frac{1}{6}(1 + 0)(2 + 0) = \frac{1}{3}.$$

Now try the following exercise, making use of the identity

$$1^3 + 2^3 + \dots + n^3 = \frac{1}{4}n^2(n+1)^2.$$

**Exercise 1.1**

Let  $f(x) = x^3$ , where  $x \in [0, 1]$ . Show that for

$$P_n = \{[0, 1/n], [1/n, 2/n], \dots, [(n-1)/n, 1]\},$$

we have

$$R(f, P_n) = \frac{1}{4}(1 + 1/n)^2,$$

and determine  $\lim_{n \rightarrow \infty} R(f, P_n)$ .

The Riemann sums  $R(f, P_n)$  of Example 1.1 approximate the area under the graph of  $y = x^2$  between 0 and 1. The approximation improves as  $n$  increases, and we expect the limiting value  $\frac{1}{3}$  to actually be the area under the graph. However, to be sure that this limit gives us a sensible value, we should check that  $R(f, P_n) \rightarrow \frac{1}{3}$  for *any* sequence  $(P_n)$  of partitions of  $[0, 1]$  such that  $\|P_n\| \rightarrow 0$ . The following important theorem, for which we omit the proof, provides this check.

### Theorem 1.1

Let  $f: [a, b] \rightarrow \mathbb{R}$  be a continuous function. Then there is a real number  $A$  such that

$$\lim_{n \rightarrow \infty} R(f, P_n) = A,$$

for any sequence  $(P_n)$  of partitions of  $[a, b]$  such that  $\|P_n\| \rightarrow 0$ .

We can now define the Riemann integral of a continuous function.

### Definition

Let  $f: [a, b] \rightarrow \mathbb{R}$  be a continuous function, where  $a < b$ . The value  $A$  determined by Theorem 1.1 is called the **Riemann integral** of  $f$  over  $[a, b]$ , and it is denoted by

$$\int_a^b f(x) dx.$$

The theorem tells us that to calculate the Riemann integral of  $f$  over  $[a, b]$ , we can make *any* choice of partitions  $(P_n)$  for which  $\|P_n\| \rightarrow 0$  and calculate  $\lim_{n \rightarrow \infty} R(f, P_n)$ . Thus the calculation of Example 1.1 really does demonstrate that

$$\int_0^1 x^2 dx = \frac{1}{3}.$$

We define the Riemann integral  $\int_a^b f(x) dx$  when  $a \geq b$ , as follows.

### Definitions

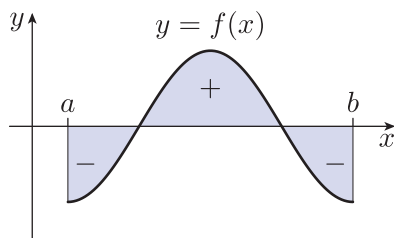
Let  $f$  be a continuous real function.

If  $a > b$ , and  $[b, a]$  is contained in the domain of  $f$ , then we define

$$\int_a^b f(x) dx = - \int_b^a f(x) dx.$$

Also, for values of  $a$  in the domain of  $f$ , we define

$$\int_a^a f(x) dx = 0.$$



**Figure 1.7** Signed area determined by the graph of  $y = f(x)$  between  $a$  and  $b$

As we have discussed, for a continuous real function  $f$  that takes only *positive* values on  $[a, b]$ , where  $a < b$ , the Riemann integral

$$\int_a^b f(x) dx$$

measures the area under the graph of  $y = f(x)$  between  $a$  and  $b$ . If we no longer require  $f$  to be positive, then the integral still has a geometric meaning: it measures the *signed area* of the set between the curve  $y = f(x)$ , the  $x$ -axis and the vertical lines  $x = a$  and  $x = b$ , where we count parts of the set above the  $x$ -axis as having positive area, and parts of the set below the  $x$ -axis as having negative area, as illustrated in Figure 1.7.

## 1.2 Properties of the Riemann integral

In practice we do not usually calculate integrals by looking at partitions, but instead use a powerful theorem known as the Fundamental Theorem of Calculus, which allows us to think of integration and differentiation as inverse processes.

To state the theorem, we need the notion of a **primitive** of a continuous real function  $f: [a, b] \rightarrow \mathbb{R}$ ; this is a real function  $F$  that is differentiable on  $[a, b]$  with derivative equal to  $f$ , that is, the function  $F$  satisfies  $F'(x) = f(x)$ , for all  $x \in [a, b]$ . A primitive of a function is not unique, because if  $F$  is a primitive of  $f$ , then so is the function with rule  $F(x) + c$ , for any constant  $c$ .

### Theorem 1.2 Fundamental Theorem of Calculus

Let  $f: [a, b] \rightarrow \mathbb{R}$  be a continuous function. If  $F$  is a primitive of  $f$ , then the Riemann integral of  $f$  over  $[a, b]$  exists and is given by

$$\int_a^b f(x) dx = F(b) - F(a).$$

We denote  $F(b) - F(a)$  by  $[F(x)]_a^b$ .

For example, a primitive of  $f(x) = x^2$  is  $F(x) = x^3/3$ , so

$$\int_0^1 x^2 dx = \left[ \frac{x^3}{3} \right]_0^1 = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3},$$

which agrees with our earlier calculation using Riemann sums.

The Riemann integral has a number of useful properties.

### Theorem 1.3 Properties of the Riemann integral

Let  $f$  and  $g$  be real functions that are continuous on the interval  $[a, b]$ .

(a) **Sum Rule** 
$$\int_a^b (f(x) + g(x)) dx = \int_a^b f(x) dx + \int_a^b g(x) dx.$$

(b) **Multiple Rule**  $\int_a^b \lambda f(x) dx = \lambda \int_a^b f(x) dx$ , for  $\lambda \in \mathbb{R}$ .

(c) **Additivity Rule**

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx, \quad \text{for } a \leq c \leq b.$$

(d) **Substitution Rule** If  $g$  is differentiable on  $[a, b]$  and its derivative  $g'$  is continuous on  $[a, b]$ , and if  $f$  is continuous on  $\{g(x) : a \leq x \leq b\}$ , then

$$\int_a^b f(g(x))g'(x) dx = \int_{g(a)}^{g(b)} f(t) dt.$$

(e) **Integration by Parts** If  $f$  and  $g$  are differentiable on  $[a, b]$  and their derivatives  $f'$  and  $g'$  are continuous on  $[a, b]$ , then

$$\int_a^b f'(x)g(x) dx = [f(x)g(x)]_a^b - \int_a^b f(x)g'(x) dx.$$

(f) **Monotonicity Inequality** If  $f(x) \leq g(x)$  for each  $x \in [a, b]$ , then

$$\int_a^b f(x) dx \leq \int_a^b g(x) dx.$$

(g) **Modulus Inequality**  $\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx.$

The first five properties are probably familiar to you and we have stated them only for reference. The last two inequalities may be less familiar. The Monotonicity Inequality, illustrated in Figure 1.8, states that if you replace  $f$  by a greater function  $g$ , then the integral increases.

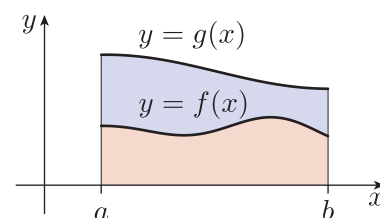
The Modulus Inequality, illustrated in Figure 1.9, says that the modulus of the integral of  $f$  over  $[a, b]$  (a non-negative number) is less than or equal to the integral of the modulus of  $f$  over  $[a, b]$  (another non-negative number). If  $f$  is positive, then these two numbers are equal, but if  $f$  takes negative values, then at least part of the signed area between  $y = f(x)$  and the  $x$ -axis is negative, so the first number is less than the second.

## Exercise 1.2

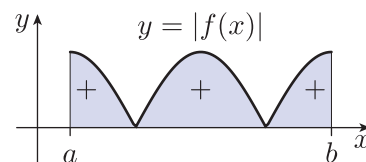
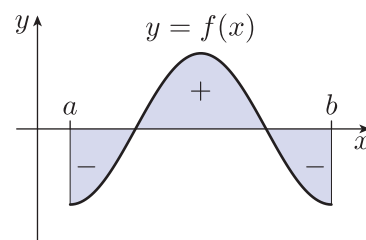
Use the Monotonicity Inequality and the fact that

$$e^{-x} \leq e^{-x^2} \leq \frac{1}{1+x^2}, \quad \text{for } 0 \leq x \leq 1,$$

to estimate  $\int_0^1 e^{-x^2} dx$  from above and below.



**Figure 1.8** Monotonicity Inequality



**Figure 1.9** Modulus Inequality

## 1.3 Introducing complex integration

We come now to the central theme of this unit – integrating complex functions. Informed by the discussion in the Introduction, we should expect that the integral of a continuous complex function  $f$  from one point  $\alpha$  to another point  $\beta$  in the complex plane may depend on the path that we choose to take from  $\alpha$  to  $\beta$ . So it is necessary to first choose a smooth path  $\Gamma : \gamma(t)$  ( $t \in [a, b]$ ) such that  $\gamma(a) = \alpha$  and  $\gamma(b) = \beta$  (see Figure 1.10), and then we will define the integral of  $f$  along this smooth path, denoting the resulting quantity by

$$\int_{\Gamma} f(z) dz.$$

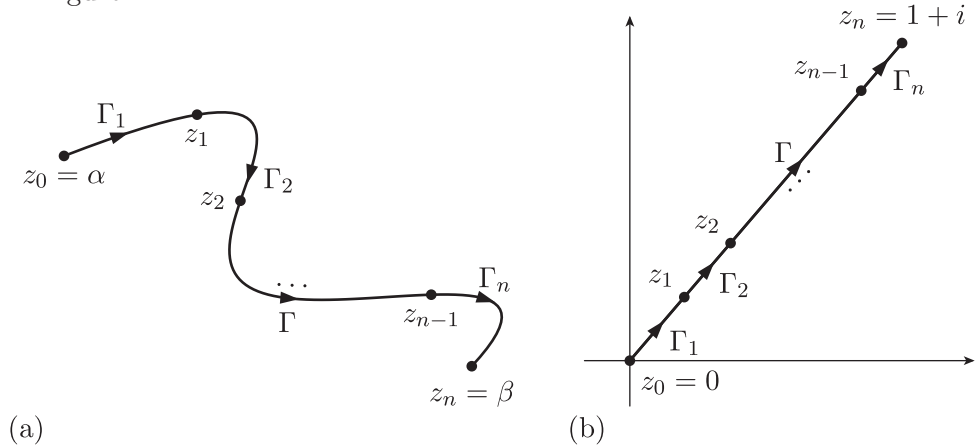
There are two ways to achieve this goal.

One method is to imitate the approach of Subsection 1.1, as follows.

- Choose a *partition* of the path  $\Gamma$  into subpaths

$$P = \{\Gamma_1, \Gamma_2, \dots, \Gamma_n\},$$

determined by points  $\alpha = z_0, z_1, \dots, z_n = \beta$ , such as those illustrated in Figure 1.11.



**Figure 1.11** (a) A partition of  $\Gamma$  (b) A partition of the line segment from 0 to  $1 + i$

- Define a complex *Riemann sum*

$$R(f, P) = \sum_{k=1}^n f(z_k) \delta z_k,$$

where  $\delta z_k = z_k - z_{k-1}$ , for  $k = 1, 2, \dots, n$ , and define

$$\|P\| = \max\{|\delta z_1|, |\delta z_2|, \dots, |\delta z_n|\}.$$

- Define the complex integral  $\int_{\Gamma} f(z) dz$  to be

$$\lim_{n \rightarrow \infty} R(f, P_n),$$

where  $(P_n)$  is any sequence of partitions of  $\Gamma$  for which  $\|P_n\| \rightarrow 0$  as  $n \rightarrow \infty$ .

It can be shown (although it is quite hard to do so) that this limit exists when  $f$  is continuous, and that it is independent of the choice of partitions of  $\Gamma$ . Thus we have defined the integral of a continuous complex function. We can then develop the standard properties of integrals, such as the Additivity Rule and the Combination Rules, by imitating the discussion of the real Riemann integral.

The second, quicker, method is to define a complex integral in terms of two real integrals. To do this, we use a parametrisation  $\gamma: [a, b] \rightarrow \mathbb{C}$  of the smooth path  $\Gamma$ , where  $\gamma(a) = \alpha$  and  $\gamma(b) = \beta$ . Any set of parameter values

$$\{t_0, t_1, \dots, t_n : a = t_0 < t_1 < \dots < t_n = b\}$$

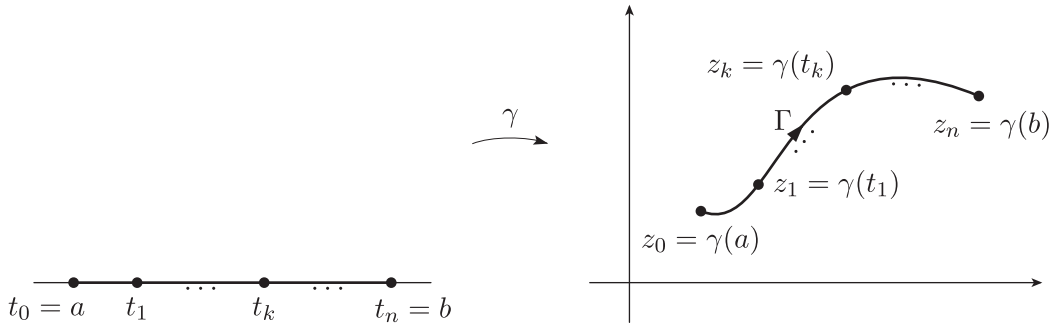
yields a partition

$$P = \{\Gamma_1, \Gamma_2, \dots, \Gamma_n\}$$

of  $\Gamma$ , where  $\Gamma_k$  is the subpath of  $\Gamma$  that joins  $z_{k-1} = \gamma(t_{k-1})$  to  $z_k = \gamma(t_k)$ , for  $k = 1, 2, \dots, n$ . We can then define the complex Riemann sum

$$R(f, P) = \sum_{k=1}^n f(z_k) \delta z_k,$$

where  $\delta z_k = z_k - z_{k-1}$ , for  $k = 1, 2, \dots, n$ ; see Figure 1.12.



**Figure 1.12** A partition of  $\Gamma$  induced by the parameter values  $t_0, t_1, \dots, t_n$

Notice that

$$\delta z_k = z_k - z_{k-1} = \gamma(t_k) - \gamma(t_{k-1}).$$

Hence, if  $t_k$  is close to  $t_{k-1}$ , then, to a good approximation,

$$\gamma'(t_k) \approx \frac{\gamma(t_k) - \gamma(t_{k-1})}{t_k - t_{k-1}} = \frac{\delta z_k}{\delta t_k},$$

where  $\delta t_k = t_k - t_{k-1}$ , so

$$\delta z_k \approx \gamma'(t_k) \delta t_k.$$

Thus if  $\max\{\delta t_1, \delta t_2, \dots, \delta t_n\}$  is small, then, to a good approximation,

$$R(f, P) = \sum_{k=1}^n f(z_k) \delta z_k \approx \sum_{k=1}^n f(\gamma(t_k)) \gamma'(t_k) \delta t_k.$$

The expression on the right has the form of a Riemann sum for the integral

$$\int_a^b f(\gamma(t)) \gamma'(t) dt. \quad (1.1)$$

Here the integrand

$$t \mapsto f(\gamma(t)) \gamma'(t) \quad (t \in [a, b])$$

is a *complex*-valued function of a *real* variable. We have defined integrals of only real functions so far, but if we split  $f(\gamma(t)) \gamma'(t)$  into its real and imaginary parts  $u(t) + iv(t)$ , then the integral (1.1) can be written as

$$\int_a^b f(\gamma(t)) \gamma'(t) dt = \int_a^b u(t) dt + i \int_a^b v(t) dt,$$

which is a combination of two real integrals. We then define the integral of  $f$  along  $\Gamma$  by the formula

$$\int_{\Gamma} f(z) dz = \int_a^b f(\gamma(t)) \gamma'(t) dt. \quad (1.2)$$

It can be shown that both of these methods for defining the integral of a continuous complex function  $f$  along a smooth path  $\Gamma$  give the same value for

$$\int_{\Gamma} f(z) dz.$$

In the next section we will develop properties of complex integrals, and there we will use formula (1.2) for the *definition* of the integral of a complex function  $f$  along a path  $\Gamma$ .

### History of complex integration

The first significant steps in the development of *real* integration came in the seventeenth century with the work of a number of European mathematicians. Notable among this group was the French lawyer and mathematician Pierre de Fermat (1601–1665), who found areas under curves of the form  $y = ax^n$ , for  $n$  an integer (possibly negative), using partitions and arguments involving infinitesimals.

A major breakthrough was the discovery of calculus made independently by the English mathematician and scientist Isaac Newton (1642–1727) and the German philosopher and mathematician Gottfried Wilhelm Leibniz (1646–1716). They observed that differentiation and integration are inverse processes, a fact encapsulated in the Fundamental Theorem of Calculus.

Towards the end of the eighteenth century, mathematicians began to consider integrating complex functions.

Two pioneers in this endeavour were Leonhard Euler and Pierre-Simon Laplace, both of whom you encountered in Book A. They were mainly concerned with manipulating complex integrals in order to evaluate difficult real integrals such as

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi \quad \text{and} \quad \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}.$$

Some of their methods were similar to those that you will meet later in the module (we return to these two particular integrals in Book C).

However, it was through the work of Augustin-Louis Cauchy, whom you also met in Book A, that complex integration began to assume the form that is now used in complex analysis. Cauchy's first paper on complex integrals in 1814 treated complex integrals as purely algebraic objects; it was only much later that he came to properly appreciate their geometric significance.

By the mid to late nineteenth century, mathematicians began to consider how to expand the theory of integration to deal with functions that are not continuous. The first rigorous theory of integration to do this was put forward by Riemann in 1854. The Riemann integral was followed by a number of other formal definitions of integration, some equivalent to Riemann's, and some more general, such as *Lebesgue integration*, named after the French mathematician Henri Lebesgue (1875–1941).

## 2 Integrating complex functions

After working through this section, you should be able to:

- define the integral of a continuous function along a smooth path, and evaluate such integrals
- explain what is meant by a *contour*, define the (*contour*) *integral* of a continuous function along a contour, and evaluate such integrals
- define the *reverse contour* of a given contour, and state and use the Reverse Contour Theorem.

### 2.1 Integration along a smooth path

Motivated by the discussion of the preceding section, we make the following definition of the integral of a complex function (which uses the concept of a smooth path, from Subsection 4.1 of Unit A4).

**Definition**

Let  $\Gamma : \gamma(t)$  ( $t \in [a, b]$ ) be a smooth path in  $\mathbb{C}$ , and let  $f$  be a function that is continuous on  $\Gamma$ . Then the **integral of  $f$  along the path  $\Gamma$** , denoted by  $\int_{\Gamma} f(z) dz$ , is

$$\int_{\Gamma} f(z) dz = \int_a^b f(\gamma(t)) \gamma'(t) dt.$$

The integral is evaluated by splitting  $f(\gamma(t)) \gamma'(t)$  into its real and imaginary parts  $u(t) = \operatorname{Re}(f(\gamma(t)) \gamma'(t))$  and  $v(t) = \operatorname{Im}(f(\gamma(t)) \gamma'(t))$ , and evaluating the resulting pair of real integrals,

$$\int_a^b f(\gamma(t)) \gamma'(t) dt = \int_a^b u(t) dt + i \int_a^b v(t) dt.$$

**Remarks**

1. Since  $f$  is continuous on  $\Gamma$  and  $\gamma$  is a smooth parametrisation, the functions  $t \mapsto f(\gamma(t))$  and  $t \mapsto \gamma'(t)$  are both continuous on  $[a, b]$ , so the function  $t \mapsto f(\gamma(t)) \gamma'(t)$  is continuous on  $[a, b]$ . It follows that the real functions  $u$  and  $v$  are continuous on  $[a, b]$ , and hence

$$\int_a^b u(t) dt \quad \text{and} \quad \int_a^b v(t) dt$$

exist, so  $\int_a^b f(\gamma(t)) \gamma'(t) dt$  also exists.

2. An important special case is when  $\gamma(t) = t$  ( $t \in [a, b]$ ), so  $\Gamma$  is the real line segment from  $a$  to  $b$ . Since  $\gamma'(t) = 1$ , we see that  $\int_{\Gamma} f(z) dz$  equals

$$\int_a^b f(t) dt = \int_a^b u(t) dt + i \int_a^b v(t) dt,$$

where  $u = \operatorname{Re} f$  and  $v = \operatorname{Im} f$ . This equation is a formula for the integral of a *complex* function over a real interval.

3. An alternative notation for  $\int_{\Gamma} f(z) dz$  is  $\int_{\Gamma} f$ .
4. If the path of integration  $\Gamma$  has a standard parametrisation  $\gamma$  (see Subsection 2.2 of Unit A2), then, unless otherwise stated, we use  $\gamma$  in the evaluation of the integral of  $f$  along  $\Gamma$ .
5. To help to remember the formula used to define  $\int_{\Gamma} f(z) dz$ , notice that it can be obtained by ‘substituting’

$$z = \gamma(t), \quad dz = \gamma'(t) dt.$$

We consider  $dz = \gamma'(t) dt$  to be a shorthand for  $\frac{dz}{dt} = \gamma'(t)$ .

The following examples demonstrate how to evaluate integrals along paths. In each case, we follow the convention of Remark 3 and use the standard parametrisation of the path.

### Example 2.1

Evaluate

$$\int_{\Gamma} z^2 dz,$$

where  $\Gamma$  is the line segment from 0 to  $1 + i$ .

### Solution

Here  $f(z) = z^2$ , and we use the standard parametrisation

$$\gamma(t) = (1 + i)t \quad (t \in [0, 1])$$

of  $\Gamma$ , which satisfies  $\gamma'(t) = 1 + i$ .

Then  $f(\gamma(t)) = ((1 + i)t)^2$ , so

$$\begin{aligned} \int_{\Gamma} z^2 dz &= \int_0^1 f(\gamma(t)) \gamma'(t) dt \\ &= \int_0^1 ((1 + i)t)^2 (1 + i) dt \\ &= \int_0^1 2it^2(1 + i) dt \\ &= \int_0^1 (-2 + 2i)t^2 dt \\ &= -2 \int_0^1 t^2 dt + 2i \int_0^1 t^2 dt \\ &= (-2 + 2i) \int_0^1 t^2 dt \\ &= (-2 + 2i) \left[ \frac{1}{3} t^3 \right]_0^1 \\ &= -\frac{2}{3} + \frac{2}{3}i. \end{aligned}$$

You need not include every line of working of Example 2.1 if you do not need to. Here is another example.

### Example 2.2

Evaluate

$$\int_{\Gamma} \bar{z} dz,$$

where  $\Gamma$  is the line segment from 0 to  $1 + i$ .

**Solution**

Here  $f(z) = \bar{z}$ , and again we use the standard parametrisation

$$\gamma(t) = (1 + i)t \quad (t \in [0, 1])$$

of  $\Gamma$ , which satisfies  $\gamma'(t) = 1 + i$ . Then

$$f(\gamma(t)) = \overline{(1 + i)t} = (1 - i)t,$$

so

$$\begin{aligned} \int_{\Gamma} \bar{z} dz &= \int_0^1 (1 - i)t \times (1 + i) dt \\ &= \int_0^1 2t dt \\ &= [t^2]_0^1 \\ &= 1. \end{aligned}$$

We set out our solution to the next example using the observation and notation of Remark 4.

**Example 2.3**

Evaluate

$$\int_{\Gamma} \frac{1}{z} dz,$$

where  $\Gamma$  is the unit circle  $\{z : |z| = 1\}$ .

**Solution**

Here  $f(z) = 1/z$ , and we use the standard parametrisation

$$\gamma(t) = e^{it} \quad (t \in [0, 2\pi])$$

of  $\Gamma$ . Then  $z = e^{it}$ ,  $1/z = e^{-it}$  and  $dz = ie^{it} dt$ . Hence

$$\begin{aligned} \int_{\Gamma} \frac{1}{z} dz &= \int_0^{2\pi} e^{-it} \times ie^{it} dt \\ &= i \int_0^{2\pi} 1 dt \\ &= 2\pi i. \end{aligned}$$

Sometimes when evaluating integrals we will use the alternative notation of Example 2.3 instead of the notation of Examples 2.1 and 2.2; both notations are commonly used in complex analysis.

In the examples above, we used the standard parametrisation in each case. The following exercise suggests that the value of the integral is not affected by the choice of parametrisation.

### Exercise 2.1

- (a) Verify that the result of Example 2.2 is unchanged if we use the smooth parametrisation

$$\gamma(t) = 2(1+i)t \quad (t \in [0, \tfrac{1}{2}]).$$

- (b) Verify that the result of Example 2.3 is unchanged if we use the smooth parametrisation

$$\gamma(t) = e^{3it} \quad (t \in [0, 2\pi/3]).$$

The reason why we have obtained the same values in Exercise 2.1 as those in Examples 2.2 and 2.3 is because of the following theorem.

### Theorem 2.1

Let  $\gamma_1: [a_1, b_1] \rightarrow \mathbb{C}$  and  $\gamma_2: [a_2, b_2] \rightarrow \mathbb{C}$  be two smooth parametrisations of paths with the same image set  $\Gamma$ , and let  $f$  be a function that is continuous on  $\Gamma$ . Then

$$\int_{\Gamma} f(z) dz$$

does not depend on which parametrisation  $\gamma_1$  or  $\gamma_2$  is used.

**Proof** We prove the theorem when  $\gamma_1$  and  $\gamma_2$  are both one-to-one functions onto the set  $\Gamma$  (so the paths do not intersect themselves). The general result can be proved by splitting the path into parts on which  $\gamma_1$  and  $\gamma_2$  are one-to-one; we omit the details.

Since  $\gamma_1$  and  $\gamma_2$  are one-to-one functions, and both are differentiable with continuous derivatives that are never zero, we can define another function

$$h(t) = \gamma_2^{-1}(\gamma_1(t)) \quad (t \in [a_1, b_1]),$$

which itself is differentiable with a continuous derivative (using the Inverse Function Rule, Theorem 3.2 of Unit A4, for  $\gamma_2^{-1}$ ). This is a one-to-one function from  $[a_1, b_1]$  to  $[a_2, b_2]$  that satisfies  $h(a_1) = a_2$  and  $h(b_1) = b_2$ .

Next, observe that  $\gamma_1(t) = \gamma_2(h(t))$ , so  $\gamma_1'(t) = \gamma_2'(h(t))h'(t)$ , by the Chain Rule. Therefore

$$\begin{aligned} \int_{a_1}^{b_1} f(\gamma_1(t)) \gamma_1'(t) dt &= \int_{a_1}^{b_1} f(\gamma_2(h(t))) \gamma_2'(h(t)) h'(t) dt \\ &= \int_{h(a_1)}^{h(b_1)} f(\gamma_2(s)) \gamma_2'(s) ds \\ &= \int_{a_2}^{b_2} f(\gamma_2(s)) \gamma_2'(s) ds, \end{aligned}$$

where we have applied the real substitution  $s = h(t)$ ,  $ds = h'(t) dt$ . Hence the integral of  $f$  along  $\Gamma$  does not depend on which parametrisation  $\gamma_1$  or  $\gamma_2$  is used. ■

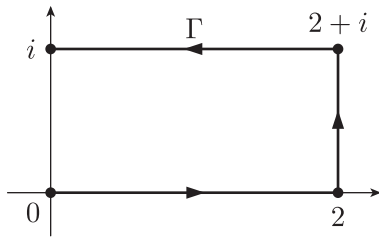
In practical terms, this theorem allows you to choose any convenient smooth parametrisation when evaluating a complex integral along a given path. We will see how this can be helpful in the next subsection.

For further practice in integration, try the following exercise.

### Exercise 2.2

Evaluate the following integrals.

- (a)  $\int_{\Gamma} \operatorname{Re} z \, dz$ , where  $\Gamma$  is the line segment from 0 to  $1 + 2i$ .
- (b)  $\int_{\Gamma} \frac{1}{(z - \alpha)^2} \, dz$ , where  $\Gamma$  is the circle with centre  $\alpha$  and radius  $r$ .



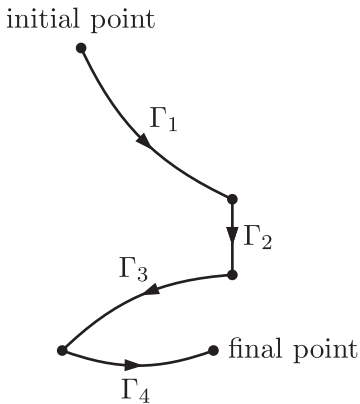
**Figure 2.1** A path  $\Gamma$  from 0 to  $i$

## 2.2 Integration along a contour

Consider the path  $\Gamma$  from 0 to  $i$  in Figure 2.1, with parametrisation  $\gamma: [0, 3] \rightarrow \mathbb{C}$  given by

$$\gamma(t) = \begin{cases} 2t, & 0 \leq t \leq 1, \\ 2 + i(t-1), & 1 \leq t \leq 2, \\ 2 + i - 2(t-2), & 2 \leq t \leq 3. \end{cases}$$

This path is not smooth, because  $\gamma$  is not differentiable at  $t = 1$  or  $t = 2$ . However,  $\Gamma$  can be split into three smooth straight-line paths, joined end to end. This leads to the idea of a *contour*: it is simply what we get when we place a finite number of smooth paths end to end.



**Figure 2.2** The contour  $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4$

### Definitions

A **contour**  $\Gamma$  is a path that can be subdivided into a finite number of smooth paths  $\Gamma_1, \Gamma_2, \dots, \Gamma_n$  joined end to end. The order of these constituent smooth paths is indicated by writing

$$\Gamma = \Gamma_1 + \Gamma_2 + \dots + \Gamma_n.$$

The **initial point** of  $\Gamma$  is the initial point of  $\Gamma_1$ , and the **final point** of  $\Gamma$  is the final point of  $\Gamma_n$ .

The definition of a contour is illustrated in Figure 2.2.

As an example, the contour  $\Gamma$  in Figure 2.1 can be written as  $\Gamma_1 + \Gamma_2 + \Gamma_3$ , where  $\Gamma_1$ ,  $\Gamma_2$  and  $\Gamma_3$  are smooth paths with smooth parametrisations

$$\begin{aligned} \gamma_1(t) &= 2t & (t \in [0, 1]), \\ \gamma_2(t) &= 2 + i(t-1) & (t \in [1, 2]), \\ \gamma_3(t) &= 2 + i - 2(t-2) & (t \in [2, 3]). \end{aligned} \tag{2.1}$$

Now, we have seen how to integrate a continuous function along a smooth path. It is natural to extend this definition to contours, by splitting the contour into smooth paths and integrating along each in turn. We formalise this idea in the following definition.

### Definition

Let  $\Gamma = \Gamma_1 + \Gamma_2 + \cdots + \Gamma_n$  be a contour, and let  $f$  be a function that is continuous on  $\Gamma$ . Then the (**contour**) **integral of  $f$  along  $\Gamma$** , denoted by  $\int_{\Gamma} f(z) dz$ , is

$$\int_{\Gamma} f(z) dz = \int_{\Gamma_1} f(z) dz + \int_{\Gamma_2} f(z) dz + \cdots + \int_{\Gamma_n} f(z) dz.$$

### Remarks

1. It is clear that a contour can be split into smooth paths in many different ways. Fortunately, all such splittings lead to the same value for the contour integral. We omit the proof of this result, as it is straightforward but tedious.
2. When evaluating an integral along a contour  $\Gamma = \Gamma_1 + \Gamma_2 + \cdots + \Gamma_n$ , we often consider each smooth path  $\Gamma_1, \Gamma_2, \dots, \Gamma_n$  separately, using a convenient parametrisation in each case. For example, consider the contour  $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3$  of Figure 2.1. To evaluate a contour integral of the form

$$\int_{\Gamma} f(z) dz = \int_{\Gamma_1} f(z) dz + \int_{\Gamma_2} f(z) dz + \int_{\Gamma_3} f(z) dz,$$

we can use the smooth parametrisations (2.1) of  $\Gamma_1$ ,  $\Gamma_2$  and  $\Gamma_3$ , or we could use another convenient choice of parametrisations, such as

$$\begin{aligned} \gamma_1(t) &= t & (t \in [0, 2]), \\ \gamma_2(t) &= 2 + it & (t \in [0, 1]), \\ \gamma_3(t) &= 2 + i - t & (t \in [0, 2]). \end{aligned}$$

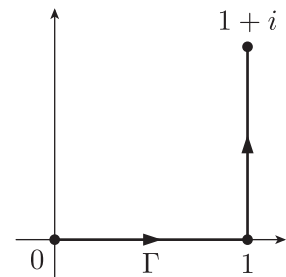
3. The alternative notation  $\int_{\Gamma} f$  is sometimes used for contour integrals when the omission of the integration variable  $z$  will cause no confusion.

### Example 2.4

Evaluate

$$\int_{\Gamma} z^2 dz,$$

where  $\Gamma$  is the contour shown in Figure 2.3.



**Figure 2.3** A contour  $\Gamma$  from 0 to  $1 + i$

**Solution**

We split  $\Gamma$  into two smooth paths  $\Gamma = \Gamma_1 + \Gamma_2$ , where  $\Gamma_1$  is the line segment from 0 to 1 with parametrisation  $\gamma_1(t) = t$  ( $t \in [0, 1]$ ), and  $\Gamma_2$  is the line segment from 1 to  $1 + i$ , with parametrisation  $\gamma_2(t) = 1 + it$  ( $t \in [0, 1]$ ). Then

$$\begin{aligned}
 \int_{\Gamma} z^2 dz &= \int_{\Gamma_1} z^2 dz + \int_{\Gamma_2} z^2 dz \\
 &= \int_0^1 t^2 dt + \int_0^1 (1 + it)^2 i dt \\
 &= \int_0^1 t^2 dt + \int_0^1 (-2t + i - it^2) dt \\
 &= \int_0^1 (t^2 - 2t) dt + i \int_0^1 (1 - t^2) dt \\
 &= \left[ \frac{1}{3}t^3 - t^2 \right]_0^1 + i \left[ t - \frac{1}{3}t^3 \right]_0^1 \\
 &= \left( \frac{1}{3} - 1 \right) + i \left( 1 - \frac{1}{3} \right) \\
 &= -\frac{2}{3} + \frac{2}{3}i.
 \end{aligned}$$

Notice that this answer is the same as that obtained in Example 2.1 for

$$\int_{\Gamma} z^2 dz,$$

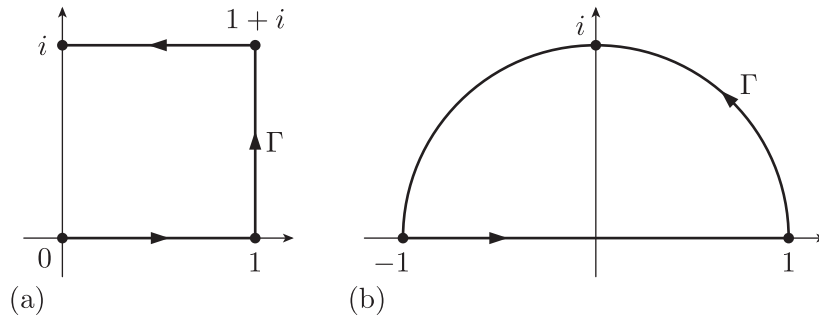
where  $\Gamma$  is the line segment from 0 to  $1 + i$ . The reason for this will become clear when we get to Theorem 3.2, the Contour Independence Theorem.

**Exercise 2.3**

Evaluate

$$\int_{\Gamma} \bar{z} dz$$

for each of the following contours  $\Gamma$ .



In part (b) the contour consists of a line segment and a semicircle, traversed once anticlockwise. Take  $-1$  to be the initial (and final) point of this contour.

We conclude this subsection by stating some rules for combining contour integrals. To prove them, we split the contour  $\Gamma$  into constituent smooth paths, and use the Sum Rule and Multiple Rule for real integration given in Theorem 1.3 to prove the results for each path. We omit the details.

### Theorem 2.2 Combination Rules for Contour Integrals

Let  $\Gamma$  be a contour, and let  $f$  and  $g$  be functions that are continuous on  $\Gamma$ .

(a) **Sum Rule**  $\int_{\Gamma} (f(z) + g(z)) dz = \int_{\Gamma} f(z) dz + \int_{\Gamma} g(z) dz.$

(b) **Multiple Rule**  $\int_{\Gamma} \lambda f(z) dz = \lambda \int_{\Gamma} f(z) dz, \quad \text{where } \lambda \in \mathbb{C}.$

## 2.3 Reverse paths and contours

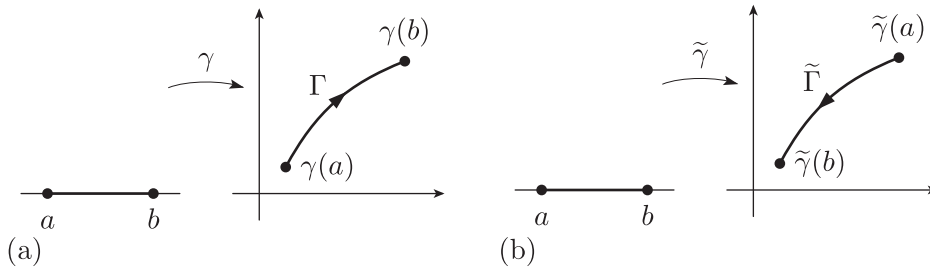
We now introduce the concept of the *reverse path* (some texts use the name *opposite path*) of a smooth path  $\Gamma$ . This is simply the path we obtain by traversing the original path in the opposite direction, starting from the final point of the original path and finishing at the initial point of the original path. In order to define the reverse path formally, we use the fact that as  $t$  increases from  $a$  to  $b$ , so  $a + b - t$  decreases from  $b$  to  $a$ .

### Definition

Let  $\Gamma : \gamma(t)$  ( $t \in [a, b]$ ) be a smooth path. Then the **reverse path** of  $\Gamma$ , denoted by  $\tilde{\Gamma}$ , is the path with parametrisation  $\tilde{\gamma}$ , where

$$\tilde{\gamma}(t) = \gamma(a + b - t) \quad (t \in [a, b]).$$

Note that the initial point  $\tilde{\gamma}(a)$  of  $\tilde{\Gamma}$  is the final point  $\gamma(b)$  of  $\Gamma$ , and the final point  $\tilde{\gamma}(b)$  of  $\tilde{\Gamma}$  is the initial point  $\gamma(a)$  of  $\Gamma$  (see Figure 2.4). The path  $\tilde{\Gamma}$  is smooth because  $\Gamma$  is smooth. Also note that, as *sets*,  $\Gamma$  and  $\tilde{\Gamma}$  are the same.



**Figure 2.4** (a) A smooth path  $\Gamma$  and (b) its reverse path  $\tilde{\Gamma}$

## Exercise 2.4

Write down the reverse path of the path  $\Gamma$  with parametrisation

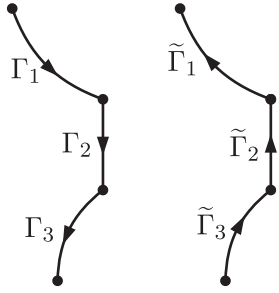
$$\gamma(t) = 2 + i - t \quad (t \in [0, 2]).$$

We can also define a *reverse contour*. This is done in the natural way – namely by reversing each of the constituent smooth paths of a contour and reversing the order in which they are traversed.

## Definition

Let  $\Gamma = \Gamma_1 + \Gamma_2 + \cdots + \Gamma_n$  be a contour. The **reverse contour**  $\tilde{\Gamma}$  of  $\Gamma$  is

$$\tilde{\Gamma} = \tilde{\Gamma}_n + \tilde{\Gamma}_{n-1} + \cdots + \tilde{\Gamma}_1.$$



**Figure 2.5** A contour  $\Gamma_1 + \Gamma_2 + \Gamma_3$  and its reverse contour  $\tilde{\Gamma}_3 + \tilde{\Gamma}_2 + \tilde{\Gamma}_1$

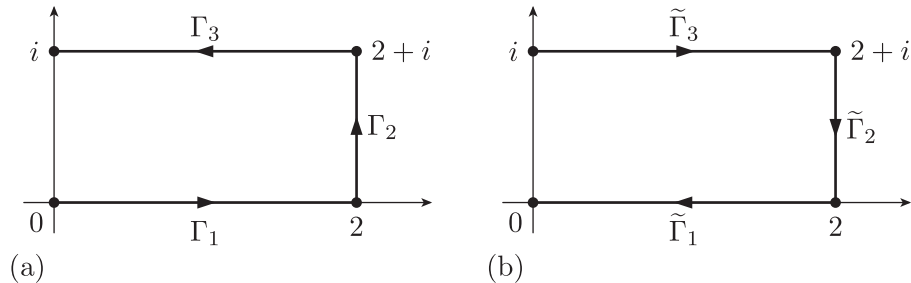
A contour and its reverse contour are illustrated in Figure 2.5.

As an example, if  $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3$  is the contour from 0 to  $i$  in Figure 2.6(a), with smooth parametrisations

$$\begin{aligned} \gamma_1(t) &= t & (t \in [0, 2]), \\ \gamma_2(t) &= 2 + it & (t \in [0, 1]), \\ \gamma_3(t) &= 2 + i - t & (t \in [0, 2]), \end{aligned}$$

then  $\tilde{\Gamma} = \tilde{\Gamma}_3 + \tilde{\Gamma}_2 + \tilde{\Gamma}_1$  is the contour from  $i$  to 0 in Figure 2.6(b), with smooth parametrisations

$$\begin{aligned} \tilde{\gamma}_3(t) &= t + i & (t \in [0, 2]), \\ \tilde{\gamma}_2(t) &= 2 + i(1 - t) & (t \in [0, 1]), \\ \tilde{\gamma}_1(t) &= 2 - t & (t \in [0, 2]). \end{aligned}$$



**Figure 2.6** (a) The contour  $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3$  (b) The reverse contour  $\tilde{\Gamma} = \tilde{\Gamma}_3 + \tilde{\Gamma}_2 + \tilde{\Gamma}_1$

**Example 2.5**

Evaluate

$$\int_{\tilde{\Gamma}} \bar{z} dz,$$

where  $\tilde{\Gamma}$  is the reverse path of the line segment  $\Gamma$  from 0 to  $1 + i$ .

**Solution**

We use the standard parametrisation

$$\gamma(t) = (1 + i)t \quad (t \in [0, 1])$$

of  $\Gamma$ . For the reverse path  $\tilde{\Gamma}$ , the corresponding parametrisation is

$$\tilde{\gamma}(t) = \gamma(1 - t) = (1 + i)(1 - t) \quad (t \in [0, 1]).$$

Then  $\tilde{\gamma}'(t) = -(1 + i)$ , so we substitute

$$z = (1 + i)(1 - t), \quad \bar{z} = (1 - i)(1 - t) \quad \text{and} \quad dz = -(1 + i) dt$$

to give

$$\begin{aligned} \int_{\tilde{\Gamma}} \bar{z} dz &= - \int_0^1 (1 - i)(1 - t) \times (1 + i) dt \\ &= - \int_0^1 2(1 - t) dt \\ &= -[2t - t^2]_0^1 = -1. \end{aligned}$$

In Example 2.2 we saw that

$$\int_{\Gamma} \bar{z} dz = 1,$$

which is the negative of the value  $-1$  that we obtained in Example 2.5. This illustrates the general result that if we integrate a function along a reverse contour  $\tilde{\Gamma}$ , then the answer is the negative of the integral of the function along  $\Gamma$ .

**Theorem 2.3 Reverse Contour Theorem**

Let  $\Gamma$  be a contour, and let  $f$  be a function that is continuous on  $\Gamma$ . Then the integral of  $f$  along the reverse contour  $\tilde{\Gamma}$  of  $\Gamma$  satisfies

$$\int_{\tilde{\Gamma}} f(z) dz = - \int_{\Gamma} f(z) dz.$$

**Proof** The proof is in two parts. We first prove the result in the case when  $\Gamma$  is a smooth path, and then extend the proof to contours.

- (a) Let  $\Gamma : \gamma(t)$  ( $t \in [a, b]$ ) be a smooth path. Then the parametrisation of  $\tilde{\Gamma}$  is

$$\tilde{\gamma}(t) = \gamma(a + b - t) \quad (t \in [a, b]).$$

It follows that  $\tilde{\gamma}'(t) = -\gamma'(a + b - t)$ , by the Chain Rule, so

$$\begin{aligned} \int_{\tilde{\Gamma}} f(z) dz &= \int_a^b f(\tilde{\gamma}(t)) \tilde{\gamma}'(t) dt \\ &= \int_a^b f(\gamma(a + b - t))(-\gamma'(a + b - t)) dt \\ &= \int_b^a f(\gamma(s)) \gamma'(s) ds \\ &= - \int_{\Gamma} f(z) dz, \end{aligned}$$

where, in the second-to-last line, we have made the real substitution

$$s = a + b - t, \quad ds = -dt.$$

- (b) To extend the proof to a general contour  $\Gamma$ , we argue as follows.

Let  $\Gamma = \Gamma_1 + \Gamma_2 + \cdots + \Gamma_n$ , for smooth paths  $\Gamma_1, \Gamma_2, \dots, \Gamma_n$ . Then

$$\tilde{\Gamma} = \tilde{\Gamma}_n + \tilde{\Gamma}_{n-1} + \cdots + \tilde{\Gamma}_1,$$

and we can apply part (a) to see that

$$\begin{aligned} \int_{\tilde{\Gamma}} f &= \int_{\tilde{\Gamma}_n} f + \int_{\tilde{\Gamma}_{n-1}} f + \cdots + \int_{\tilde{\Gamma}_1} f \\ &= - \int_{\Gamma_n} f - \int_{\Gamma_{n-1}} f - \cdots - \int_{\Gamma_1} f \\ &= - \left( \int_{\Gamma_n} f + \int_{\Gamma_{n-1}} f + \cdots + \int_{\Gamma_1} f \right) \\ &= - \int_{\Gamma} f. \end{aligned}$$



In Example 2.3 we saw that

$$\int_{\Gamma} \frac{1}{z} dz = 2\pi i,$$

where  $\Gamma$  is the unit circle  $\{z : |z| = 1\}$ . The next exercise asks you to check Theorem 2.3 for this contour integral.

### Exercise 2.5

Verify that

$$\int_{\tilde{\Gamma}} \frac{1}{z} dz = -2\pi i,$$

where  $\Gamma$  is the unit circle.

## Further exercises

### Exercise 2.6

Evaluate the following integrals (using the standard parametrisation of the path  $\Gamma$  in each case).

(a) (i)  $\int_{\Gamma} z \, dz$ , (ii)  $\int_{\Gamma} \operatorname{Im} z \, dz$ , (iii)  $\int_{\Gamma} \bar{z} \, dz$ ,  
where  $\Gamma$  is the line segment from 1 to  $i$ .

(b) (i)  $\int_{\Gamma} \bar{z} \, dz$ , (ii)  $\int_{\Gamma} z^2 \, dz$ ,  
where  $\Gamma$  is the unit circle  $\{z : |z| = 1\}$ .

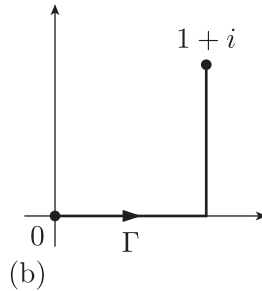
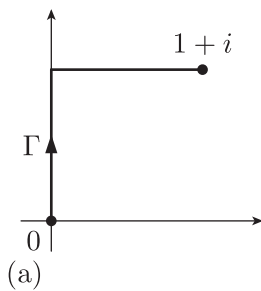
(c) (i)  $\int_{\Gamma} \frac{1}{z} \, dz$ , (ii)  $\int_{\Gamma} |z| \, dz$ ,  
where  $\Gamma$  is the upper half of the circle with centre 0 and radius 2 traversed from 2 to  $-2$ .

### Exercise 2.7

Evaluate

$$\int_{\Gamma} \operatorname{Re} z \, dz$$

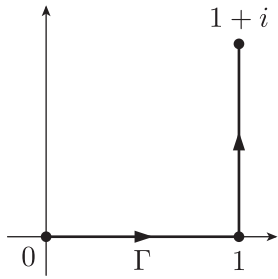
for each of the following contours  $\Gamma$  from 0 to  $1 + i$ .



## 3 Evaluating contour integrals

After working through this section, you should be able to:

- state and use the Fundamental Theorem of Calculus for contour integrals
- state and use the Contour Independence Theorem
- use the technique of Integration by Parts
- state and use the Closed Contour Theorem, the Grid Path Theorem, the Zero Derivative Theorem and the Paving Theorem.



**Figure 3.1** A contour  $\Gamma$  from 0 to  $1+i$

### 3.1 The Fundamental Theorem of Calculus

In Example 2.4 we saw that

$$\int_{\Gamma} z^2 dz = -\frac{2}{3} + \frac{2}{3}i,$$

where  $\Gamma$  is the contour shown in Figure 3.1. Our method was to write down a smooth parametrisation for each of the two line segments, replace  $z$  in the integral by these parametrisations, and then integrate. It is, however, tempting to approach this integral as you would a corresponding real integral and write

$$\begin{aligned} \int_{\Gamma} z^2 dz &= \left[ \frac{1}{3} z^3 \right]_0^{1+i} \\ &= \frac{1}{3} (1+i)^3 - \frac{1}{3} \times 0^3 \\ &= \frac{1}{3} (1 + 3i + 3i^2 + i^3) \\ &= -\frac{2}{3} + \frac{2}{3}i. \end{aligned}$$

The Fundamental Theorem of Calculus for contour integrals tells us that this method of evaluation is permissible under certain conditions. Before stating it, we need the idea of a *primitive* of a complex function, which is defined in a similar way to the primitive of a real function (Subsection 1.2).

#### Definition

Let  $f$  and  $F$  be functions defined on a region  $\mathcal{R}$ . Then  $F$  is a **primitive of  $f$  on  $\mathcal{R}$**  if  $F$  is analytic on  $\mathcal{R}$  and

$$F'(z) = f(z), \quad \text{for all } z \in \mathcal{R}.$$

The function  $F$  is also called an *antiderivative* or *indefinite integral* of  $f$  on  $\mathcal{R}$ .

For example,  $F(z) = \frac{1}{3}z^3$  is a primitive of  $f(z) = z^2$  on  $\mathbb{C}$ , since  $F$  is analytic on  $\mathbb{C}$  and  $F'(z) = z^2$ , for all  $z \in \mathbb{C}$ . Another primitive is  $F(z) = \frac{1}{3}z^3 + 2i$ ; indeed, *any* function of the form  $F(z) = \frac{1}{3}z^3 + c$ , where  $c \in \mathbb{C}$ , is a primitive of  $f$  on  $\mathbb{C}$ .

#### Exercise 3.1

Write down a primitive  $F$  of each of the following functions  $f$  on the given region  $\mathcal{R}$ .

- (a)  $f(z) = e^{3iz}$ ,  $\mathcal{R} = \mathbb{C}$
- (b)  $f(z) = (1 + iz)^{-2}$ ,  $\mathcal{R} = \mathbb{C} - \{i\}$
- (c)  $f(z) = z^{-1}$ ,  $\mathcal{R} = \{z : \operatorname{Re} z > 0\}$

We now state the Fundamental Theorem of Calculus for contour integrals, which gives us a quick way of evaluating a contour integral of a function

with a primitive that we can determine. The theorem will be proved later in this subsection.

### Theorem 3.1 Fundamental Theorem of Calculus

Let  $f$  be a function that is continuous and has a primitive  $F$  on a region  $\mathcal{R}$ , and let  $\Gamma$  be a contour in  $\mathcal{R}$  with initial point  $\alpha$  and final point  $\beta$ . Then

$$\int_{\Gamma} f(z) dz = F(\beta) - F(\alpha).$$

We often use the notation

$$[F(z)]_{\alpha}^{\beta} = F(\beta) - F(\alpha).$$

Some texts write  $F(z)|_{\alpha}^{\beta}$  instead of  $[F(z)]_{\alpha}^{\beta}$ .

For an example of the use of the Fundamental Theorem of Calculus, observe that if  $f(z) = z^2$ , then  $f$  is continuous on  $\mathbb{C}$  and has a primitive  $F(z) = \frac{1}{3}z^3$  there. Hence, for the contour  $\Gamma$  in Figure 3.1, we *can* write

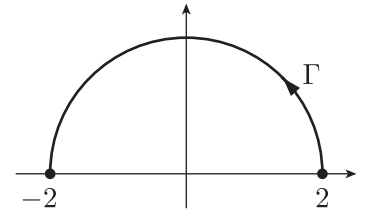
$$\int_{\Gamma} z^2 dz = \left[ \frac{1}{3}z^3 \right]_0^{1+i} = \frac{1}{3}(1+i)^3 - \frac{1}{3} \times 0^3 = -\frac{2}{3} + \frac{2}{3}i.$$

### Exercise 3.2

Use the Fundamental Theorem of Calculus to evaluate

$$\int_{\Gamma} e^{3iz} dz,$$

where  $\Gamma$  is the semicircular path shown in Figure 3.2.



**Figure 3.2** A semicircular path  $\Gamma$  from 2 to  $-2$

You have seen that

$$\int_{\Gamma} z^2 dz = -\frac{2}{3} + \frac{2}{3}i$$

both when  $\Gamma$  is the contour in Figure 3.1 and also when  $\Gamma$  is the line segment from 0 to  $1+i$  (see Example 2.1). This is not a coincidence: in fact, it is a particular case of the following important consequence of the Fundamental Theorem of Calculus.

### Theorem 3.2 Contour Independence Theorem

Let  $f$  be a function that is continuous and has a primitive  $F$  on a region  $\mathcal{R}$ , and let  $\Gamma_1$  and  $\Gamma_2$  be contours in  $\mathcal{R}$  with the same initial point  $\alpha$  and the same final point  $\beta$ . Then

$$\int_{\Gamma_1} f(z) dz = \int_{\Gamma_2} f(z) dz.$$

**Proof** By the Fundamental Theorem of Calculus for contour integrals, the value of each of these integrals is  $F(\beta) - F(\alpha)$ . ■

The idea that a contour integral may, under suitable hypotheses, depend only on the endpoints of the contour (and not on the contour itself) will prove to have great significance in the next unit.

### Exercise 3.3

Use the Fundamental Theorem of Calculus to evaluate the following integrals.

- (a)  $\int_{\Gamma} e^{-\pi z} dz$ , where  $\Gamma$  is any contour from  $-i$  to  $i$ .
- (b)  $\int_{\Gamma} (3z - 1)^2 dz$ , where  $\Gamma$  is any contour from  $2$  to  $2i + \frac{1}{3}$ .
- (c)  $\int_{\Gamma} \sinh z dz$ , where  $\Gamma$  is any contour from  $i$  to  $1$ .
- (d)  $\int_{\Gamma} e^{\sin z} \cos z dz$ , where  $\Gamma$  is any contour from  $0$  to  $\pi/2$ .
- (e)  $\int_{\Gamma} \frac{\sin z}{\cos^2 z} dz$ , where  $\Gamma$  is any contour from  $0$  to  $\pi$  lying in  $\mathbb{C} - \{(n + \frac{1}{2})\pi : n \in \mathbb{Z}\}$ .

Next we give a version of Integration by Parts for contour integrals, which we will need later in the module.

### Theorem 3.3 Integration by Parts

Let  $f$  and  $g$  be functions that are analytic on a region  $\mathcal{R}$ , and suppose that  $f'$  and  $g'$  are continuous on  $\mathcal{R}$ . Let  $\Gamma$  be a contour in  $\mathcal{R}$  with initial point  $\alpha$  and final point  $\beta$ . Then

$$\int_{\Gamma} f(z)g'(z) dz = [f(z)g(z)]_{\alpha}^{\beta} - \int_{\Gamma} f'(z)g(z) dz.$$

**Proof** Let  $H(z) = f(z)g(z)$  and  $h(z) = f'(z)g(z) + f(z)g'(z)$ . Then  $h$  is continuous on  $\mathcal{R}$ , by hypothesis. Also,  $h$  has primitive  $H$ , since  $H$  is analytic on  $\mathcal{R}$  and

$$H'(z) = h(z),$$

by the Product Rule for differentiation. It follows from the Fundamental Theorem of Calculus that

$$\int_{\Gamma} h(z) dz = [H(z)]_{\alpha}^{\beta};$$

that is,

$$\int_{\Gamma} (f'(z)g(z) + f(z)g'(z)) dz = [f(z)g(z)]_{\alpha}^{\beta}.$$

Using the Sum Rule (Theorem 2.2(a)) and rearranging the resulting equation, we obtain

$$\int_{\Gamma} f(z)g'(z) dz = [f(z)g(z)]_{\alpha}^{\beta} - \int_{\Gamma} f'(z)g(z) dz,$$

as required. ■

### Example 3.1

Use Integration by Parts to evaluate

$$\int_{\Gamma} ze^{2z} dz,$$

where  $\Gamma$  is any contour from 0 to  $\pi i$ .

### Solution

We take  $f(z) = z$ ,  $g(z) = \frac{1}{2}e^{2z}$  and  $\mathcal{R} = \mathbb{C}$ . Then  $f$  and  $g$  are analytic on  $\mathcal{R}$ , and  $f'(z) = 1$  and  $g'(z) = e^{2z}$  are continuous on  $\mathcal{R}$ .

Integrating by parts, we obtain

$$\begin{aligned} \int_{\Gamma} ze^{2z} dz &= [z \times \tfrac{1}{2}e^{2z}]_0^{\pi i} - \int_{\Gamma} 1 \times \tfrac{1}{2}e^{2z} dz \\ &= (\pi i \times \tfrac{1}{2}e^{2\pi i} - 0) - [\tfrac{1}{4}e^{2z}]_0^{\pi i} \\ &= \tfrac{1}{2}\pi i - (\tfrac{1}{4} - \tfrac{1}{4}) \\ &= \tfrac{1}{2}\pi i. \end{aligned}$$

### Exercise 3.4

Use Integration by Parts to evaluate the following integrals.

- (a)  $\int_{\Gamma} z \cosh z dz$ , where  $\Gamma$  is any contour from 0 to  $\pi i$ .
- (b)  $\int_{\Gamma} \operatorname{Log} z dz$ , where  $\Gamma$  is any contour from 1 to  $i$  lying in the cut plane  $\mathbb{C} - \{x \in \mathbb{R} : x \leq 0\}$ .
- (Hint: For part (b), take  $f(z) = \operatorname{Log} z$  and  $g(z) = z$ .)

The Fundamental Theorem of Calculus is a useful tool when the function  $f$  being integrated has an easily determined primitive  $F$ . However, if the function  $f$  has no primitive, or if we are unable to find one, then we have to resort to the definition of an integral and use parametrisation. For example, we cannot use the Fundamental Theorem of Calculus to evaluate

$$\int_{\Gamma} \bar{z} dz$$

along any contour, since the function  $f(z) = \bar{z}$  has no primitive on any region.

To see why this is so, suppose that  $f$  is a function that is defined on a region in the complex plane. We observe that *if  $f$  is not differentiable, then  $f$  has no primitive  $F$* . This is because, as we will see in the next unit, any differentiable complex function can be differentiated as many times as we like. Thus, if  $f$  has a primitive  $F$ , then  $F$  is differentiable with  $F' = f$ . Hence  $f$  is also differentiable.

It follows that we cannot use the Fundamental Theorem of Calculus to evaluate integrals of non-differentiable functions such as

$$z \mapsto \bar{z}, \quad z \mapsto \operatorname{Re} z, \quad z \mapsto \operatorname{Im} z \quad \text{and} \quad z \mapsto |z|.$$

We conclude this subsection by proving the Fundamental Theorem of Calculus.

**Proof of the Fundamental Theorem of Calculus** The proof is in two parts. We first prove the result in the case when  $\Gamma$  is a smooth path, and then extend the proof to contours.

(a) Let  $\Gamma : \gamma(t)$  ( $t \in [a, b]$ ) be a smooth path. Then

$$\begin{aligned} \int_{\Gamma} f(z) dz &= \int_a^b f(\gamma(t)) \gamma'(t) dt \\ &= \int_a^b F'(\gamma(t)) \gamma'(t) dt \\ &= \int_a^b (F \circ \gamma)'(t) dt, \end{aligned}$$

by the Chain Rule. Now, if we write  $(F \circ \gamma)(t)$  as a sum of its real and imaginary parts  $u(t) + iv(t)$ , then

$$\int_a^b (F \circ \gamma)'(t) dt = \int_a^b u'(t) dt + i \int_a^b v'(t) dt.$$

The Fundamental Theorem of Calculus for *real* integrals (Theorem 1.2) tells us that

$$\int_a^b u'(t) dt = u(b) - u(a) \quad \text{and} \quad \int_a^b v'(t) dt = v(b) - v(a).$$

Hence

$$\int_{\Gamma} f(z) dz = (u(b) - u(a)) + i(v(b) - v(a)) = F(\beta) - F(\alpha),$$

since  $\beta = \gamma(b)$  and  $\alpha = \gamma(a)$ .

(b) To extend the proof to a general contour  $\Gamma$  with initial point  $\alpha$  and final point  $\beta$ , we argue as follows.

Let  $\Gamma = \Gamma_1 + \Gamma_2 + \cdots + \Gamma_n$ , for smooth paths  $\Gamma_1, \Gamma_2, \dots, \Gamma_n$ , and let the initial and final points of  $\Gamma_k$  be  $\alpha_k$  and  $\beta_k$ , for  $k = 1, 2, \dots, n$ .

Then

$$\alpha_1 = \alpha, \quad \alpha_2 = \beta_1, \quad \dots, \quad \alpha_n = \beta_{n-1}, \quad \beta_n = \beta.$$

By part (a),

$$\int_{\Gamma_k} f(z) dz = F(\beta_k) - F(\alpha_k) = F(\beta_k) - F(\beta_{k-1}),$$

for  $k = 1, 2, \dots, n$  (where  $\beta_0 = \alpha$ ). Hence

$$\begin{aligned} \int_{\Gamma} f(z) dz &= \int_{\Gamma_1} f(z) dz + \int_{\Gamma_2} f(z) dz + \cdots + \int_{\Gamma_n} f(z) dz \\ &= (F(\beta_1) - F(\beta_0)) + \cdots + (F(\beta_n) - F(\beta_{n-1})) \\ &= F(\beta_n) - F(\beta_0) \\ &= F(\beta) - F(\alpha). \end{aligned}$$

■

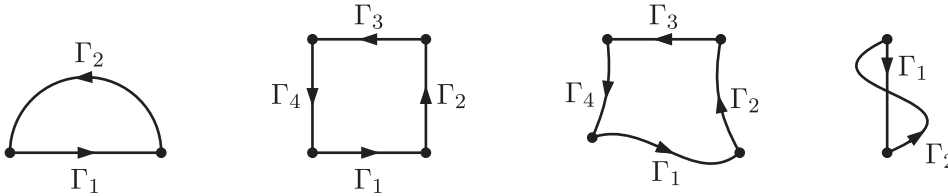
## 3.2 Closed paths and grid paths

The Fundamental Theorem of Calculus has a number of important theoretical consequences, which we now describe. The first concerns integration around *closed contours* (where the adjective ‘closed’ has a different meaning to that in the phrase ‘closed set’).

### Definition

A path or contour  $\Gamma$  is **closed** if its initial and final points coincide.

For example, any circle is a closed contour, as are the contours in Figure 3.3.



**Figure 3.3** Closed contours

We remarked earlier (after the definition of contour integrals in Subsection 2.2) that any method of splitting a contour  $\Gamma$  into constituent smooth paths leads to the same value for a given contour integral along  $\Gamma$ . Using this fact, it can be shown that if  $\Gamma$  is a closed contour, then the value of any contour integral along  $\Gamma$  does not depend on the choice of initial point of  $\Gamma$ .

The following result about closed contours is a forerunner of Cauchy’s Theorem – a major result which appears in the next unit.

### Theorem 3.4 Closed Contour Theorem

Let  $f$  be a function that is continuous and has a primitive  $F$  on a region  $\mathcal{R}$ . Then

$$\int_{\Gamma} f(z) dz = 0,$$

for any closed contour  $\Gamma$  in  $\mathcal{R}$ .

**Proof** Since  $\Gamma$  is closed, its initial and final points  $\alpha$  and  $\beta$  are equal. It follows from the Fundamental Theorem of Calculus that

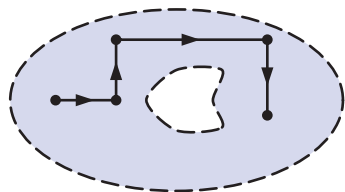
$$\int_{\Gamma} f(z) dz = F(\beta) - F(\alpha) = 0.$$

### Exercise 3.5

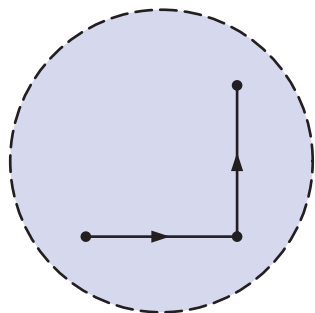
Use the Closed Contour Theorem to prove the following statements.

- (a)  $\int_{\Gamma} 1/z dz = 0$ , where  $\Gamma$  is the circle with centre  $1 + i$  and radius 1.  
 (b)  $\int_{\Gamma} 1/z^2 dz = 0$ , where  $\Gamma$  is the unit circle  $\{z : |z| = 1\}$ .

Next we use the Fundamental Theorem of Calculus to prove that a function that is analytic on a region  $\mathcal{R}$  and has derivative zero throughout  $\mathcal{R}$  is constant on  $\mathcal{R}$ . To prove this result in its most general form, we need a further property of regions. Recall that any two points in a region  $\mathcal{R}$  can be joined by a path in  $\mathcal{R}$ , because  $\mathcal{R}$  is connected. In fact, such a path can always be chosen to be a contour of the following simple type (along which we can integrate).



**Figure 3.4** A grid path in a region



**Figure 3.5** Two points joined by a grid path in an open disc

### Definition

A **grid path** is a contour each of whose constituent smooth paths is a line segment parallel to either the real axis or the imaginary axis.

For example, the contour shown in Figure 3.4 is a grid path.

Regions have the following important property.

### Theorem 3.5 Grid Path Theorem

Any two points in a region  $\mathcal{R}$  can be joined by a grid path in  $\mathcal{R}$ .

The assertion of the Grid Path Theorem is obvious if  $\mathcal{R}$  is an open disc, where at most two line segments are required (Figure 3.5), but it is more difficult to prove for a *general* region. We postpone the proof until after the next result, which demonstrates how useful grid paths can be.

### Theorem 3.6 Zero Derivative Theorem

Let  $f$  be a function that is analytic on a region  $\mathcal{R}$ , and let  $f'(z) = 0$ , for all  $z$  in  $\mathcal{R}$ . Then  $f$  is constant on  $\mathcal{R}$ .

**Proof** Let  $\alpha$  and  $\beta$  be any two points in  $\mathcal{R}$ , and let  $\Gamma$  be any grid path in  $\mathcal{R}$  with initial point  $\alpha$  and final point  $\beta$ . Since  $f'(z) = 0$ , for all  $z$  in  $\mathcal{R}$ , it follows from the Fundamental Theorem of Calculus that

$$f(\beta) - f(\alpha) = \int_{\Gamma} f'(z) dz = \int_{\Gamma} 0 dz = 0.$$

Thus  $f(\alpha) = f(\beta)$ , and hence  $f$  is constant on  $\mathcal{R}$ . ■

The next exercise gives an important consequence of the Zero Derivative Theorem, which will be needed later in the module.

### Exercise 3.6

Prove that if  $F_1$  and  $F_2$  are both primitives of a function  $f$  on a region  $\mathcal{R}$ , then

$$F_1(z) = F_2(z) + c, \quad \text{for all } z \text{ in } \mathcal{R},$$

where  $c$  is a complex constant.

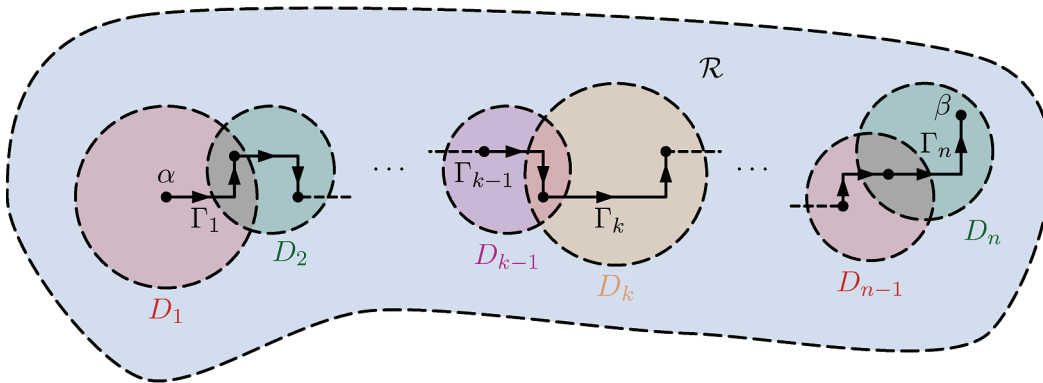
## 3.3 Proof of the Grid Path Theorem

The proofs in this subsection are technical, and the methods are not central to later units, so you may wish to skip the details on a first reading.

We have remarked that the assertion of the Grid Path Theorem is obvious if the region  $\mathcal{R}$  is an open disc (see Figure 3.5). To prove the theorem in general, we make use of this special case and show that we can construct a grid path joining two points  $\alpha$  and  $\beta$  in  $\mathcal{R}$  of the form

$$\Gamma_1 + \Gamma_2 + \cdots + \Gamma_n,$$

where each  $\Gamma_k$ ,  $k = 1, 2, \dots, n$ , is a grid path in an open disc  $D_k \subseteq \mathcal{R}$  that joins two points in  $D_k$ . Figure 3.6 illustrates this construction, and shows the need for a finite sequence of overlapping discs  $D_1, D_2, \dots, D_n$  that link  $\alpha$  to  $\beta$ .



**Figure 3.6** Open discs  $D_1, D_2, \dots, D_n$  covering a grid path  $\Gamma_1 + \Gamma_2 + \cdots + \Gamma_n$

To show that such a sequence of discs always exists, we introduce the following definition and theorem, the Paving Theorem. A proof of the Grid Path Theorem then follows almost immediately.

### Definitions

A **paving** of a path  $\Gamma : \gamma(t)$  ( $t \in [a, b]$ ) is a finite sequence

$$D_1, D_2, \dots, D_n$$

of open discs such that there are numbers  $t_1, t_2, \dots, t_n$  satisfying

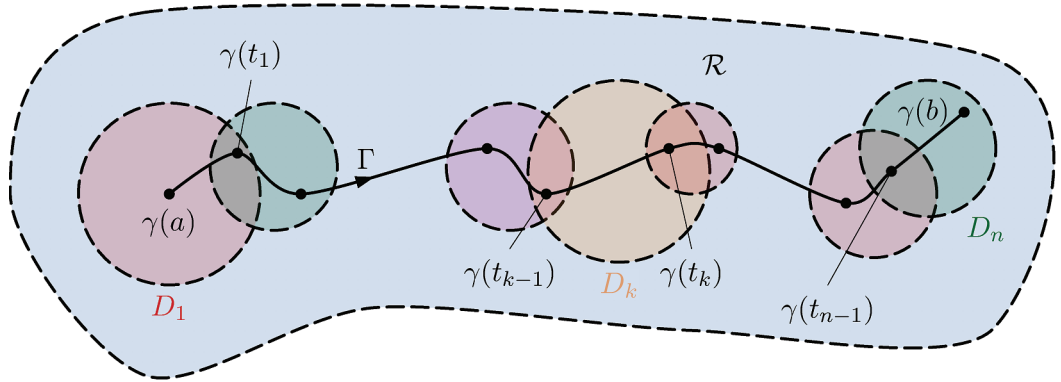
$$a = t_0 < t_1 < \dots < t_n = b$$

for which

$$\gamma([t_{k-1}, t_k]) \subseteq D_k, \quad k = 1, 2, \dots, n.$$

We say that the discs  $D_k$  **pave**  $\Gamma$ .

We now state the Paving Theorem, which is illustrated in Figure 3.7.



**Figure 3.7** Open discs  $D_1, D_2, \dots, D_n$  paving a path  $\Gamma$  in a region  $\mathcal{R}$

### Theorem 3.7 Paving Theorem

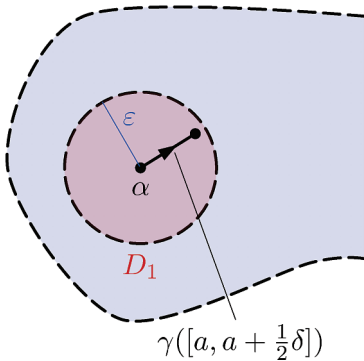
Any path  $\Gamma : \gamma(t)$  ( $t \in [a, b]$ ) lying in a region  $\mathcal{R}$  can be paved by open discs  $D_1, D_2, \dots, D_n$  lying in  $\mathcal{R}$  such that

$$\bigcup_{k=1}^n D_k \subseteq \mathcal{R}.$$

**Proof** Let  $\alpha = \gamma(a)$  and  $\beta = \gamma(b)$  be the initial and final points of  $\Gamma$ . Because  $\mathcal{R}$  is open, there is an open disc  $\{z : |z - \alpha| < \varepsilon\}$ , say, lying in  $\mathcal{R}$ . Since  $\gamma$  is continuous, there is a  $\delta > 0$  such that

$$t \in [a, a + \delta) \implies |\gamma(t) - \gamma(a)| < \varepsilon.$$

Hence  $\gamma([a, a + \frac{1}{2}\delta]) \subseteq \{z : |z - \alpha| < \varepsilon\}$ , so it is possible to start paving  $\Gamma$  with the disc  $D_1 = \{z : |z - \alpha| < \varepsilon\}$ ; see Figure 3.8.



**Figure 3.8** The first disc  $D_1$  of a paving

Now consider the set  $S$  of real numbers  $s \in [a, b]$  such that the path determined by  $\gamma(t)$  ( $t \in [a, s]$ ) can be paved. We have just seen that  $a + \frac{1}{2}\delta \in S$ , and we want to show that  $b \in S$ . Clearly,  $S \subseteq [a, b]$  and  $S$  has the property that if  $s \in S$  and  $a \leq t < s$ , then  $t \in S$  also. Hence  $S$  is an interval with left-hand endpoint  $a$ , and its right-hand endpoint is some number  $c$  that satisfies  $a < c \leq b$ . We will show that  $c \in S$  and then that  $c = b$ , thus completing the proof.

Because  $\mathcal{R}$  is open, there is a disc  $D' = \{z : |z - \gamma(c)| < \varepsilon'\}$ , say, lying in  $\mathcal{R}$ . Since  $\gamma$  is continuous, there is a  $\delta' > 0$  such that, for  $t \in [a, b]$ ,

$$\begin{aligned} |t - c| < \delta' &\implies |\gamma(t) - \gamma(c)| < \varepsilon' \\ &\implies \gamma(t) \in D'. \end{aligned} \quad (3.1)$$

Now, if  $s \in [a, b]$  and  $c - \delta' < s < c$ , then  $s \in S$ , so the path determined by  $\gamma([a, s])$  can be paved. Since  $\gamma(s) \in D'$ , we can add  $D'$  to this paving, to deduce, by statement (3.1), that

$$\gamma([a, t]) \text{ can be paved, for } t \in [a, b] \text{ with } c \leq t < c + \delta'. \quad (3.2)$$

It follows that  $c \in S$  and, moreover, that  $c = b$  (since otherwise statement (3.2) would contradict the definition of  $c$ ). Hence  $b \in S$ , as required. ■

We can now prove the Grid Path Theorem.

### Theorem 3.5 Grid Path Theorem

Any two points in a region  $\mathcal{R}$  can be joined by a grid path in  $\mathcal{R}$ .

**Proof** Let  $\alpha$  and  $\beta$  be any two points in  $\mathcal{R}$ . Since  $\mathcal{R}$  is a region, it is connected, so  $\alpha$  and  $\beta$  can be joined by a path  $\Gamma : \gamma(t)$  ( $t \in [a, b]$ ). Then  $\alpha = \gamma(a)$  and  $\beta = \gamma(b)$ .

By the Paving Theorem, the path  $\Gamma$  can be paved by open discs  $D_1, D_2, \dots, D_n$  lying in  $\mathcal{R}$  such that

$$\bigcup_{k=1}^n D_k \subseteq \mathcal{R}.$$

Therefore there are numbers  $t_0, t_1, \dots, t_n$  such that

$$a = t_0 < t_1 < \dots < t_n = b,$$

and the part of the path given by  $\gamma([t_{k-1}, t_k])$  lies in the open disc  $D_k$ , for  $k = 1, 2, \dots, n$ . Since  $\gamma(t_{k-1})$  and  $\gamma(t_k)$  both lie in the disc  $D_k$ , there is a grid path  $\Gamma_k$  in  $D_k$  from one point to the other. Hence

$$\Gamma_1 + \Gamma_2 + \dots + \Gamma_n$$

forms a grid path in  $\mathcal{R}$  joining  $\alpha = \gamma(a)$  to  $\beta = \gamma(b)$ . ■

We will find a number of other similar uses for the Paving Theorem later in the module.

## Further exercises

### Exercise 3.7

For each of the following functions  $f$ , evaluate

$$\int_{\Gamma} f(z) dz,$$

where  $\Gamma$  is any contour from  $-i$  to  $i$ .

- (a)  $f(z) = 1$       (b)  $f(z) = z$       (c)  $f(z) = 5z^4 + 3iz^2$   
 (d)  $f(z) = (1 + 2iz)^9$       (e)  $f(z) = e^{-iz}$       (f)  $f(z) = \sin z$   
 (g)  $f(z) = ze^{z^2}$       (h)  $f(z) = z^3 \cosh(z^4)$       (i)  $f(z) = ze^z$

### Exercise 3.8

Evaluate the following integrals. (In each case pay special attention to the hypotheses of the theorems you use.)

- (a)  $\int_{\Gamma} \frac{1}{z} dz$ , where  $\Gamma$  is the arc of the circle  $\{z : |z| = 1\}$  from  $-i$  to  $i$  passing through 1.  
 (b)  $\int_{\Gamma} \sqrt{z} dz$ , where  $\Gamma$  is as in part (a).  
 (c)  $\int_{\Gamma} \sin^2 z dz$ , where  $\Gamma$  is the unit circle  $\{z : |z| = 1\}$ .  
 (d)  $\int_{\Gamma} \frac{1}{z^3} dz$ , where  $\Gamma$  is the circle  $\{z : |z| = 27\}$ .

(Hint: For part (c), use the identity  $\sin^2 z = \frac{1}{2}(1 - \cos 2z)$ .)

### Exercise 3.9

Construct a grid path from  $\alpha$  to  $\beta$  in the domain of the function  $\tan$ , for each of the following cases.

- (a)  $\alpha = 1, \beta = 6$       (b)  $\alpha = \frac{\pi}{2} + 2i, \beta = -\frac{3\pi}{2} - i$

## 4 Estimating contour integrals

After working through this section, you should be able to:

- calculate the *length* of a smooth path or contour
- state the Estimation Theorem and use it to obtain an upper estimate for the modulus of a given contour integral.

## 4.1 The length of a contour

It is easy to calculate the length of a line segment or a circular path, or of a contour made up from such paths. Sometimes, however, we need to find the lengths of other contours, so we must define exactly what we mean by the length of a general contour. To do this, we first define the length of a smooth path, and then define the length of a general contour to be the sum of the lengths of its constituent paths.

The definition is given below, but first we give a heuristic argument which suggests the form of the definition.

Let  $\Gamma : \gamma(t)$  ( $t \in [a, b]$ ) be a smooth path, and let  $L(\Gamma)$  be its length, which we wish to define. We now approximate  $\Gamma$  by a polygonal contour consisting of line segments joining the points

$$z_k = \gamma(t_k), \quad k = 0, 1, \dots, n,$$

where  $a = t_0 < t_1 < \dots < t_n = b$  (see Figure 4.1). An approximation for  $L(\Gamma)$  is then given by the sum of the lengths of these line segments:

$$L(\Gamma) \approx \sum_{k=1}^n |z_k - z_{k-1}| = \sum_{k=1}^n |\gamma(t_k) - \gamma(t_{k-1})|.$$

Now,  $\Gamma$  is a smooth path, so  $\gamma'(t_k)$  exists for each  $k = 1, 2, \dots, n$ , and is approximately equal to  $(\gamma(t_k) - \gamma(t_{k-1})) / (t_k - t_{k-1})$ . Hence if  $n$  is large, and the length  $t_k - t_{k-1}$  of each subinterval  $[t_{k-1}, t_k]$  is small, then

$$|\gamma(t_k) - \gamma(t_{k-1})| \approx |\gamma'(t_k)| (t_k - t_{k-1}).$$

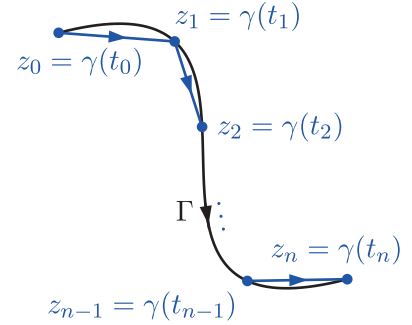
Therefore

$$L(\Gamma) \approx \sum_{k=1}^n |\gamma'(t_k)| (t_k - t_{k-1}) = \sum_{k=1}^n |\gamma'(t_k)| \delta t_k,$$

where  $\delta t_k = t_k - t_{k-1}$ . You may recognise this type of expression from the study of real integration in Section 1; it is a Riemann sum for the integral

$$\int_a^b |\gamma'(t)| dt.$$

Since  $\gamma$  is smooth, the function  $|\gamma'|$  is continuous, so this integral exists. We will take this integral as our definition of  $L(\Gamma)$ .



**Figure 4.1** Approximating a smooth path  $\Gamma$  by line segments

### Definitions

Let  $\Gamma : \gamma(t)$  ( $t \in [a, b]$ ) be a smooth path. Then the **length of the path**  $\Gamma$  is

$$L(\Gamma) = \int_a^b |\gamma'(t)| dt.$$

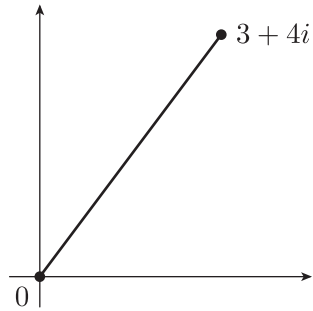
The **length of a contour** is the sum of the lengths of its constituent smooth paths.

## Remarks

1. It can be shown that the length of a contour is independent of the way that the contour is split into smooth paths, but we do not prove this.
2. Using a proof similar to that of Theorem 2.1, it is easy to check that the length of  $\Gamma$  is unchanged if  $\gamma$  is replaced by any other smooth parametrisation of  $\Gamma$ .
3. It is also easy to check that

$$L(\tilde{\Gamma}) = L(\Gamma),$$

where  $\tilde{\Gamma}$  is the reverse path of  $\Gamma$ .



**Figure 4.2** The line segment from 0 to  $3 + 4i$

## Example 4.1

Verify that the length of the line segment from 0 to  $3 + 4i$  (illustrated in Figure 4.2) is 5.

## Solution

The standard parametrisation of the given line segment is

$$\gamma(t) = (3 + 4i)t \quad (t \in [0, 1]).$$

Then  $\gamma'(t) = 3 + 4i$ , so

$$|\gamma'(t)| = \sqrt{3^2 + 4^2} = 5.$$

Thus the required length is

$$\int_0^1 5 \, dt = 5,$$

as expected.

## Exercise 4.1

- (a) Verify that the circumference of the circle with centre  $\alpha$  and radius  $r$  is  $2\pi r$ .
- (b) Find the length of the path with parametrisation

$$\gamma(t) = t + i \cosh t \quad (t \in [0, 1]).$$

In this module we will mainly use contours consisting of line segments and arcs of circles, whose lengths you knew before meeting the definition. However, the definition will be needed in theoretical work, and in examples where you do not already know the length of the path.

## 4.2 The Estimation Theorem

For a given function  $f$  that is continuous on a contour  $\Gamma$ , it may be impossible to evaluate  $\int_{\Gamma} f(z) dz$  exactly. However, we can derive a result which gives an upper estimate for the *modulus* of the integral.

This result, which we state below and prove in the next subsection, is of immense theoretical importance: indeed, many of the main proofs in complex analysis involve at least one application of it. When allied with the Residue Theorem, which you will meet in Unit C1, it also becomes an exceedingly useful technique in the *evaluation* of integrals.

### Theorem 4.1 Estimation Theorem

Let  $f$  be a function that is continuous on a contour  $\Gamma$  of length  $L$ , with

$$|f(z)| \leq M, \quad \text{for } z \in \Gamma.$$

Then

$$\left| \int_{\Gamma} f(z) dz \right| \leq ML.$$

Note that  $f$  is required to be continuous only on  $\Gamma$ .

The following example, which uses the identity  $|e^z| = e^{\operatorname{Re} z}$  from Theorem 4.1(b) of Unit A2, illustrates the use of the Estimation Theorem.

### Example 4.2

Find an upper estimate for

$$\left| \int_{\Gamma} \frac{e^z}{z} dz \right|,$$

where  $\Gamma$  is the upper half of the unit circle  $\{z : |z| = 1\}$ , which is illustrated in Figure 4.3.

### Solution

In order to apply the Estimation Theorem, we need to find the length  $L$  of  $\Gamma$ , and find an upper estimate  $M$  for  $|f(z)|$  on  $\Gamma$ , where

$$f(z) = \frac{e^z}{z}.$$

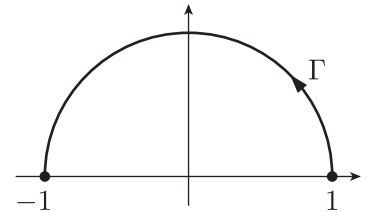
We have  $L = \pi$ , as  $\Gamma$  is a semicircle of radius 1. To find a value for  $M$ , observe that for  $z \in \Gamma$  we have

$$|z| = 1$$

and

$$|e^z| = e^{\operatorname{Re} z} \leq e^1 = e,$$

since  $\operatorname{Re} z \leq 1$ , for  $|z| = 1$ .



**Figure 4.3** A semicircular path from 1 to  $-1$

Thus

$$|f(z)| = \left| \frac{e^z}{z} \right| = \frac{|e^z|}{|z|} \leq e, \quad \text{for } z \in \Gamma,$$

so we can take  $M = e$ . Since  $f$  is continuous on  $\mathbb{C} - \{0\}$  (by the Quotient Rule for continuous functions), it is certainly continuous on  $\Gamma$ . It follows from the Estimation Theorem that

$$\left| \int_{\Gamma} \frac{e^z}{z} dz \right| \leq \pi e.$$

When applying the Estimation Theorem we seek *some* upper estimate for the modulus of the integral, so there is no unique correct answer. The aim, roughly speaking, is to find a reasonably simple estimate that is not excessively large, without too much algebraic manipulation.

### Exercise 4.2

Use the Estimation Theorem to find an upper estimate for

$$\left| \int_{\Gamma} \frac{e^{3z}}{z^2} dz \right|,$$

where  $\Gamma$  is

- (a) the circle  $\{z : |z| = 5\}$
- (b) the square contour with vertices at

$$5 + 5i, \quad -5 + 5i, \quad -5 - 5i, \quad 5 - 5i$$

(traversed once anticlockwise, with initial point  $5 + 5i$ ).

When determining a value for  $M$ , we often need to use the Triangle Inequality in one of its various forms (see Theorem 5.1 of Unit A1 and its corollary), which it is convenient to label as follows.

### Triangle Inequality

**Usual form** If  $z_1, z_2 \in \mathbb{C}$ , then

$$|z_1 + z_2| \leq |z_1| + |z_2|, \quad |z_1 - z_2| \leq |z_1| + |z_2|.$$

**Backwards form** If  $z_1, z_2 \in \mathbb{C}$ , then

$$|z_1 + z_2| \geq ||z_1| - |z_2||, \quad |z_1 - z_2| \geq ||z_1| - |z_2||.$$

The following examples illustrate the use of the Triangle Inequality in the estimation of integrals.

**Example 4.3**

Find an upper estimate for

$$\left| \int_{\Gamma} \frac{1}{z^2 + 1} dz \right|,$$

where  $\Gamma$  is the upper half of the circle  $\{z : |z| = 3\}$ .

**Solution**

The length  $L$  of  $\Gamma$  is  $3\pi$ , as  $\Gamma$  is a semicircle of radius 3. To find an upper estimate  $M$  for  $|f(z)|$  on  $\Gamma$ , where

$$f(z) = \frac{1}{z^2 + 1},$$

we need a *lower* estimate for  $|z^2 + 1|$  on  $\Gamma$ . By the Triangle Inequality (backwards form), we have

$$|z^2 + 1| \geq ||z^2| - 1| = |3^2 - 1| = 8, \quad \text{for } z \in \Gamma.$$

Thus

$$\left| \frac{1}{z^2 + 1} \right| \leq \frac{1}{8}, \quad \text{for } z \in \Gamma,$$

so we can take  $M = 1/8$ . Since  $f$  is continuous on  $\mathbb{C} - \{i, -i\}$ , it is also continuous on  $\Gamma$ , so we can apply the Estimation Theorem to obtain

$$\left| \int_{\Gamma} \frac{1}{z^2 + 1} dz \right| \leq \frac{1}{8} \times 3\pi = \frac{3\pi}{8}.$$

**Example 4.4**

Find an upper estimate for

$$\left| \int_{\Gamma} \frac{z^2 - 4z - 3}{(z^2 - 7)(z^2 + 2)} dz \right|,$$

where  $\Gamma$  is the circle  $\{z : |z| = 2\}$ .

**Solution**

The length  $L$  of  $\Gamma$  is  $4\pi$ , as  $\Gamma$  is a circle of radius 2. To find an upper estimate  $M$  for  $|f(z)|$  on  $\Gamma$ , where

$$f(z) = \frac{z^2 - 4z - 3}{(z^2 - 7)(z^2 + 2)},$$

we need an *upper* estimate for  $|z^2 - 4z - 3|$  on  $\Gamma$ , and *lower* estimates for  $|z^2 - 7|$  and  $|z^2 + 2|$  on  $\Gamma$ .

By the Triangle Inequality (usual form),

$$\begin{aligned} |z^2 - 4z - 3| &\leq |z^2| + |-4z| + |-3| \\ &= 4 + 8 + 3 = 15, \quad \text{for } z \in \Gamma, \end{aligned}$$

and by the Triangle Inequality (backwards form),

$$|z^2 - 7| \geq |2^2 - 7| = 3, \quad \text{for } z \in \Gamma,$$

and

$$|z^2 + 2| \geq |2^2 - 2| = 2, \quad \text{for } z \in \Gamma.$$

Thus

$$\left| \frac{z^2 - 4z - 3}{(z^2 - 7)(z^2 + 2)} \right| \leq \frac{15}{3 \times 2} = \frac{5}{2}, \quad \text{for } z \in \Gamma,$$

so we can take  $M = 5/2$ . Since  $f$  is continuous on  $\mathbb{C} - \{\sqrt{7}, -\sqrt{7}, i\sqrt{2}, -i\sqrt{2}\}$ , it is also continuous on  $\Gamma$ , so we can apply the Estimation Theorem to obtain

$$\left| \int_{\Gamma} \frac{z^2 - 4z - 3}{(z^2 - 7)(z^2 + 2)} dz \right| \leq \frac{5}{2} \times 4\pi = 10\pi.$$

### Exercise 4.3

Use the Estimation Theorem to find an upper estimate for

$$\left| \int_{\Gamma} \frac{3z - 4}{2z - 5} dz \right|,$$

where  $\Gamma$  is the circle  $\{z : |z| = 3\}$ .

Our final example involves both the exponential function and use of the Triangle Inequality.

### Example 4.5

Find an upper estimate for

$$\left| \int_{\Gamma} \frac{(z - 3i)e^{iz}}{z^2 + 4} dz \right|,$$

where  $\Gamma$  is the upper half of the circle  $\{z : |z| = 5\}$ .

### Solution

The length  $L$  of  $\Gamma$  is  $5\pi$ , as  $\Gamma$  is a semicircle of radius 5. To find a value for  $M$  note that, by the Triangle Inequality (both forms), we have

$$|z - 3i| \leq |z| + |-3i| = 5 + 3 = 8, \quad \text{for } z \in \Gamma,$$

and

$$|z^2 + 4| \geq |5^2 - 4| = 21, \quad \text{for } z \in \Gamma.$$

For the exponential term, we can write

$$|e^{iz}| = |e^{i(x+iy)}| = |e^{ix}| |e^{-y}| = |e^{-y}| = e^{-y}.$$

Now,  $y \geq 0$ , for  $z \in \Gamma$ , so  $e^{-y} \leq e^0 = 1$ .

Thus

$$\left| \frac{(z - 3i)e^{iz}}{z^2 + 4} \right| \leq \frac{8 \times 1}{21} = \frac{8}{21}, \quad \text{for } z \in \Gamma,$$

so we can take  $M = 8/21$ . Since  $f(z) = (z - 3i)e^{iz}/(z^2 + 4)$  is continuous on  $\mathbb{C} - \{2i, -2i\}$  (by the Composition Rule and the Combination Rules for continuous functions), it is also continuous on  $\Gamma$ , so we can apply the Estimation Theorem to obtain

$$\left| \int_{\Gamma} \frac{(z - 3i)e^{iz}}{z^2 + 4} dz \right| \leq \frac{8}{21} \times 5\pi = \frac{40\pi}{21}.$$

### Exercise 4.4

Use the Estimation Theorem to find an upper estimate for

$$\left| \int_{\Gamma} \frac{e^{2iz}}{z^2 - 9} dz \right|,$$

where  $\Gamma$  is the upper half of the circle  $\{z : |z| = 4\}$ , traversed from 4 to  $-4$ .

## 4.3 Proof of the Estimation Theorem

We now prove the Estimation Theorem. This subsection may be omitted on a first reading.

We first present a useful lemma which extends the Modulus Inequality for real integrals (Theorem 1.3(g)) from real functions to complex functions.

### Lemma 4.1

Let  $g: [a, b] \rightarrow \mathbb{C}$  be a continuous function. Then

$$\left| \int_a^b g(t) dt \right| \leq \int_a^b |g(t)| dt.$$

**Proof** Let us write the complex number  $\int_a^b g(t) dt$  in polar form, as

$$\int_a^b g(t) dt = re^{i\theta}.$$

(We cannot do this if the complex number is 0, but in that case the lemma is true anyway.) Then

$$\begin{aligned} r &= e^{-i\theta} \int_a^b g(t) dt \\ &= \int_a^b e^{-i\theta} g(t) dt \\ &= \int_a^b \operatorname{Re}(e^{-i\theta} g(t)) dt + i \int_a^b \operatorname{Im}(e^{-i\theta} g(t)) dt. \end{aligned}$$

Equating real parts, we see that

$$r = \int_a^b \operatorname{Re}(e^{-i\theta} g(t)) dt.$$

But  $\left| \int_a^b g(t) dt \right| = |re^{i\theta}| = r$ , since  $|e^{i\theta}| = 1$ , so

$$\begin{aligned} \left| \int_a^b g(t) dt \right| &= \int_a^b \operatorname{Re}(e^{-i\theta} g(t)) dt \\ &\leq \int_a^b |e^{-i\theta} g(t)| dt \\ &= \int_a^b |g(t)| dt, \end{aligned}$$

where, to obtain the second-to-last line, we used the inequality  $\operatorname{Re} z \leq |z|$  and the Monotonicity Inequality for real integrals (Theorem 1.3(f)). ■

We can now prove the Estimation Theorem, which we repeat for convenience.

#### Theorem 4.1 Estimation Theorem

Let  $f$  be a function that is continuous on a contour  $\Gamma$  of length  $L$ , with

$$|f(z)| \leq M, \quad \text{for } z \in \Gamma.$$

Then

$$\left| \int_{\Gamma} f(z) dz \right| \leq ML.$$

**Proof** The proof is in two parts. We first prove the result in the case when  $\Gamma$  is a smooth path, and then extend the proof to contours.

(a) Let  $\Gamma : \gamma(t)$  ( $t \in [a, b]$ ) be a smooth path. Then

$$\int_{\Gamma} f(z) dz = \int_a^b f(\gamma(t)) \gamma'(t) dt.$$

Applying Lemma 4.1 with  $g(t) = f(\gamma(t)) \gamma'(t)$ , we obtain

$$\begin{aligned} \left| \int_{\Gamma} f(z) dz \right| &\leq \int_a^b |f(\gamma(t)) \gamma'(t)| dt \\ &= \int_a^b |f(\gamma(t))| |\gamma'(t)| dt \\ &\leq \int_a^b M |\gamma'(t)| dt, \end{aligned}$$

using the Monotonicity Inequality for real integrals (Theorem 1.3(f)) and the fact that  $|f(z)| \leq M$ , for  $z \in \Gamma$ . Hence

$$\left| \int_{\Gamma} f(z) dz \right| \leq M \int_a^b |\gamma'(t)| dt = ML,$$

by the definition of  $L$ . This proves the result in this case.

(b) To extend the proof to a general contour  $\Gamma$ , we argue as follows.

Let  $\Gamma = \Gamma_1 + \Gamma_2 + \cdots + \Gamma_n$ , for smooth paths  $\Gamma_1, \Gamma_2, \dots, \Gamma_n$ .

Let  $L_k = L(\Gamma_k)$ , for  $k = 1, 2, \dots, n$ , so  $\Gamma$  has length

$$L = L_1 + L_2 + \cdots + L_n.$$

Then

$$\int_{\Gamma} f(z) dz = \int_{\Gamma_1} f(z) dz + \int_{\Gamma_2} f(z) dz + \cdots + \int_{\Gamma_n} f(z) dz.$$

By the Triangle Inequality and part (a), we have

$$\begin{aligned} \left| \int_{\Gamma} f \right| &\leq \left| \int_{\Gamma_1} f \right| + \left| \int_{\Gamma_2} f \right| + \cdots + \left| \int_{\Gamma_n} f \right| \\ &\leq ML_1 + ML_2 + \cdots + ML_n \\ &= M(L_1 + L_2 + \cdots + L_n) \\ &= ML. \end{aligned}$$

This completes the proof. ■

## Further exercises

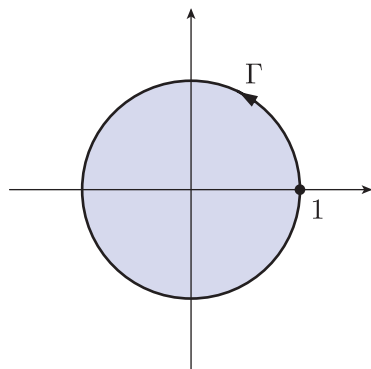
### Exercise 4.5

For each of the following functions, find an upper estimate for

$$\left| \int_{\Gamma} f(z) dz \right|,$$

where  $\Gamma$  is the circle  $\{z : |z| = 3\}$ .

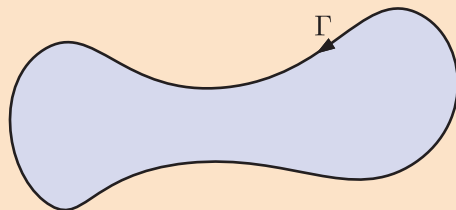
$$(a) \ f(z) = \frac{z-3}{z+2} \quad (b) \ f(z) = \frac{z^2+4}{z^2-4} \quad (c) \ f(z) = \frac{\sin z}{1+z^2}$$



**Figure 4.5** Area inside the unit circle

### Finding areas with contour integrals

We began this unit by finding areas under curves using Riemann integration, and you may have wondered whether contour integrals can also be used to find areas. More precisely, suppose that  $\Gamma$  is a closed contour that is traversed once anticlockwise and that does not intersect itself, such as the contour shown in Figure 4.4. Can we use contour integration to determine the area  $A$  inside  $\Gamma$ ?



**Figure 4.4** Area inside a closed contour

In fact, we can find this area  $A$ ; it is given by the **Area Formula**

$$A = \frac{1}{2i} \int_{\Gamma} \bar{z} dz.$$

For example, suppose that  $\Gamma$  is the unit circle, shown in Figure 4.5, traversed once anticlockwise. We can parametrise  $\Gamma$  by  $\gamma(t) = e^{it}$  ( $t \in [0, 2\pi]$ ), so  $\gamma'(t) = ie^{it}$ , and hence

$$\frac{1}{2i} \int_{\Gamma} \bar{z} dz = \frac{1}{2i} \int_0^{2\pi} e^{-it} \times ie^{it} dt = \int_0^{2\pi} \frac{1}{2} dt = \pi.$$

Since the area inside  $\Gamma$  is  $A = \pi$ , we have verified the Area Formula in this special case.

It is worth observing the contrast between the two integrals

$$\int_{\Gamma} \bar{z} dz = 2iA \quad \text{and} \quad \int_{\Gamma} z dz = 0$$

(the second integral is 0 by the Fundamental Theorem of Calculus). The crucial difference between these two integrals is that the integrand  $f(z) = z$  of the second integral is analytic on the whole of the complex plane, but the integrand  $g(z) = \bar{z}$  of the first integral is *not* analytic anywhere on the complex plane. The significance of this difference will become more apparent when you study Cauchy's Theorem in the next unit.

The Area Formula is one of many integral formulas that relate the boundary of a region to its interior. It is a special case of an important result known as **Green's Theorem**, named after the English mathematical physicist George Green (1793–1841), which is much used in engineering and physics.

# Solutions to exercises

## Solution to Exercise 1.1

Each of the  $n$  subintervals of  $P_n$  has length  $1/n$ . Therefore

$$\begin{aligned} R(f, P_n) &= \sum_{k=1}^n f\left(\frac{k}{n}\right) \times \frac{1}{n} \\ &= \sum_{k=1}^n \left(\frac{k}{n}\right)^3 \times \frac{1}{n} \\ &= \frac{1}{n^4} \sum_{k=1}^n k^3 \\ &= \frac{1}{n^4} \times \frac{1}{4} n^2 (n+1)^2 \\ &= \frac{1}{4} (1 + 1/n)^2, \end{aligned}$$

as required.

Since  $(1/n)$  is a basic null sequence, we see that

$$\lim_{n \rightarrow \infty} R(f, P_n) = \frac{1}{4} (1 + 0)^2 = \frac{1}{4}.$$

## Solution to Exercise 1.2

Since

$$e^{-x} \leq e^{-x^2} \leq \frac{1}{1+x^2}, \quad \text{for } 0 \leq x \leq 1,$$

it follows from the Monotonicity Inequality that

$$\int_0^1 e^{-x} dx \leq \int_0^1 e^{-x^2} dx \leq \int_0^1 \frac{1}{1+x^2} dx.$$

Hence

$$[-e^{-x}]_0^1 \leq \int_0^1 e^{-x^2} dx \leq [\tan^{-1} x]_0^1;$$

that is,

$$1 - e^{-1} \leq \int_0^1 e^{-x^2} dx \leq \frac{\pi}{4}.$$

Since  $0.63 < 1 - e^{-1}$  and  $\pi/4 < 0.79$ , we see that

$$0.63 < \int_0^1 e^{-x^2} dx < 0.79.$$

(In fact,  $\int_0^1 e^{-x^2} dx = 0.75$  to two decimal places.)

## Solution to Exercise 2.1

(a) Here  $\gamma(t) = 2(1+i)t$  ( $t \in [0, \frac{1}{2}]$ ). Let  $f(z) = \bar{z}$ . Then

$$f(\gamma(t)) = \overline{2(1+i)t} = 2(1-i)t,$$

and, since  $\gamma'(t) = 2(1+i)$ , we obtain

$$\begin{aligned} \int_{\Gamma} \bar{z} dz &= \int_0^{1/2} 2(1-i)t \times 2(1+i) dt \\ &= \int_0^{1/2} 8t dt \\ &= [4t^2]_0^{1/2} \\ &= 1, \end{aligned}$$

in accordance with Example 2.2.

(b) We set out this solution in a similar style to Example 2.3.

Here  $\gamma(t) = e^{3it}$  ( $t \in [0, 2\pi/3]$ ). Then

$$z = e^{3it}, \quad 1/z = e^{-3it} \quad \text{and} \quad dz = 3ie^{3it} dt.$$

Hence

$$\begin{aligned} \int_{\Gamma} \frac{1}{z} dz &= \int_0^{2\pi/3} e^{-3it} \times 3ie^{3it} dt \\ &= i \int_0^{2\pi/3} 3 dt \\ &= i [3t]_0^{2\pi/3} \\ &= 2\pi i, \end{aligned}$$

in accordance with Example 2.3.

## Solution to Exercise 2.2

(a) The standard parametrisation of  $\Gamma$  is

$$\gamma(t) = (1+2i)t \quad (t \in [0, 1]).$$

Then

$$z = (1+2i)t, \quad \operatorname{Re} z = t, \quad dz = (1+2i) dt.$$

Hence

$$\begin{aligned} \int_{\Gamma} \operatorname{Re} z dz &= \int_0^1 t \times (1+2i) dt \\ &= (1+2i) \int_0^1 t dt \\ &= (1+2i) \left[ \frac{1}{2} t^2 \right]_0^1 \\ &= \frac{1}{2} + i. \end{aligned}$$

(b) The standard parametrisation of  $\Gamma$  is

$$\gamma(t) = \alpha + re^{it} \quad (t \in [0, 2\pi]).$$

Then

$$z = \alpha + re^{it}, \quad 1/(z - \alpha)^2 = 1/(r^2 e^{2it}), \\ dz = rie^{it} dt.$$

Hence

$$\begin{aligned} \int_{\Gamma} \frac{1}{(z - \alpha)^2} dz &= \int_0^{2\pi} \frac{rie^{it}}{r^2 e^{2it}} dt \\ &= \int_0^{2\pi} \frac{i}{r} e^{-it} dt \\ &= \int_0^{2\pi} \frac{i}{r} (\cos t - i \sin t) dt \\ &= \int_0^{2\pi} \frac{1}{r} \sin t dt + i \int_0^{2\pi} \frac{1}{r} \cos t dt \\ &= \left[ -\frac{1}{r} \cos t \right]_0^{2\pi} + i \left[ \frac{1}{r} \sin t \right]_0^{2\pi} \\ &= 0 + 0i = 0. \end{aligned}$$

### Solution to Exercise 2.3

(a)  $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3$ , where  $\Gamma_1$  is the line segment from 0 to 1,  $\Gamma_2$  is the line segment from 1 to  $1 + i$ , and  $\Gamma_3$  is the line segment from  $1 + i$  to  $i$ . We choose to use the associated standard parametrisations

$$\begin{aligned} \gamma_1(t) &= t \quad (t \in [0, 1]), \\ \gamma_2(t) &= 1 + it \quad (t \in [0, 1]), \\ \gamma_3(t) &= 1 - t + i \quad (t \in [0, 1]). \end{aligned}$$

Then  $\gamma_1'(t) = 1$ ,  $\gamma_2'(t) = i$ ,  $\gamma_3'(t) = -1$ . Hence

$$\begin{aligned} \int_{\Gamma} \bar{z} dz &= \int_{\Gamma_1} \bar{z} dz + \int_{\Gamma_2} \bar{z} dz + \int_{\Gamma_3} \bar{z} dz \\ &= \int_0^1 t \times 1 dt + \int_0^1 (1 - it) \times i dt \\ &\quad + \int_0^1 (1 - t - i) \times (-1) dt \\ &= \int_0^1 (3t + 2i - 1) dt \\ &= \left[ \frac{3}{2}t^2 + (2i - 1)t \right]_0^1 \\ &= \frac{3}{2} + 2i - 1 = \frac{1}{2} + 2i. \end{aligned}$$

(b)  $\Gamma = \Gamma_1 + \Gamma_2$ , where  $\Gamma_1$  is the line segment from  $-1$  to  $1$ , and  $\Gamma_2$  is the upper half of the circle

with centre 0 from 1 to  $-1$ . We choose to use the parametrisations

$$\begin{aligned} \gamma_1(t) &= t \quad (t \in [-1, 1]), \\ \gamma_2(t) &= e^{it} \quad (t \in [0, \pi]). \end{aligned}$$

Then  $\gamma_1'(t) = 1$ ,  $\gamma_2'(t) = ie^{it}$ . Hence

$$\begin{aligned} \int_{\Gamma} \bar{z} dz &= \int_{\Gamma_1} \bar{z} dz + \int_{\Gamma_2} \bar{z} dz \\ &= \int_{-1}^1 t \times 1 dt + \int_0^{\pi} e^{-it} \times ie^{it} dt \\ &= \int_{-1}^1 t dt + i \int_0^{\pi} 1 dt \\ &= \left[ \frac{1}{2}t^2 \right]_{-1}^1 + i[t]_0^{\pi} \\ &= 0 + i\pi = \pi i. \end{aligned}$$

*Remark:* You will see an interesting interpretation of this type of integral at the very end of the unit.

### Solution to Exercise 2.4

Since  $a = 0$  and  $b = 2$ , the reverse path is  $\tilde{\Gamma} : \tilde{\gamma}(t)$  ( $t \in [0, 2]$ ), where

$$\begin{aligned} \tilde{\gamma}(t) &= \gamma(2 - t) \\ &= 2 + i - (2 - t) \\ &= t + i \quad (t \in [0, 2]). \end{aligned}$$

### Solution to Exercise 2.5

In Example 2.3 we used the parametrisation

$$\gamma(t) = e^{it} \quad (t \in [0, 2\pi]).$$

For the reverse path  $\tilde{\Gamma}$  we use the parametrisation

$$\tilde{\gamma}(t) = \gamma(2\pi - t) = e^{i(2\pi - t)} \quad (t \in [0, 2\pi]).$$

Since  $e^{2\pi i} = 1$ , we have

$$\tilde{\gamma}(t) = e^{-it} \quad (t \in [0, 2\pi]),$$

and  $\tilde{\gamma}'(t) = -ie^{-it}$ . Hence

$$\begin{aligned} \int_{\tilde{\Gamma}} \frac{1}{z} dz &= \int_0^{2\pi} \frac{1}{e^{-it}} \times (-ie^{-it}) dt \\ &= -i \int_0^{2\pi} 1 dt \\ &= -2\pi i. \end{aligned}$$

(Therefore, by Example 2.3,

$$\int_{\tilde{\Gamma}} \frac{1}{z} dz = - \int_{\Gamma} \frac{1}{z} dz.)$$

## Solution to Exercise 2.6

(a) The standard parametrisation of  $\Gamma$ , the line segment from 1 to  $i$ , is

$$\gamma(t) = 1 - t + it \quad (t \in [0, 1]);$$

hence

$$\gamma'(t) = i - 1.$$

(i) Here  $f(z) = z$ , and

$$\begin{aligned} \int_{\Gamma} z \, dz &= \int_0^1 (1 - t + it) \times (i - 1) \, dt \\ &= \int_0^1 (-1 + (1 - 2t)i) \, dt \\ &= \int_0^1 (-1) \, dt + i \int_0^1 (1 - 2t) \, dt \\ &= [-t]_0^1 + i[t - t^2]_0^1 \\ &= -1. \end{aligned}$$

(ii) Here  $f(z) = \operatorname{Im} z$ , and

$$\begin{aligned} \int_{\Gamma} \operatorname{Im} z \, dz &= \int_0^1 (\operatorname{Im}(1 - t + it)) \times (i - 1) \, dt \\ &= \int_0^1 t(i - 1) \, dt \\ &= (i - 1) \int_0^1 t \, dt \\ &= (i - 1) \left[ \frac{1}{2} t^2 \right]_0^1 \\ &= \frac{1}{2}(-1 + i). \end{aligned}$$

(Note that this integral is different from

$\operatorname{Im} \left( \int_{\Gamma} z \, dz \right)$ , which from part (a)(i) is 0.)

(iii) Here  $f(z) = \bar{z}$ , and

$$\begin{aligned} \int_{\Gamma} \bar{z} \, dz &= \int_0^1 \overline{(1 - t + it)} \times (i - 1) \, dt \\ &= \int_0^1 (1 - t - it) \times (i - 1) \, dt \\ &= \int_0^1 (-1 + 2t + i) \, dt \\ &= \int_0^1 (-1 + 2t) \, dt + i \int_0^1 1 \, dt \\ &= [-t + t^2]_0^1 + i[t]_0^1 \\ &= i. \end{aligned}$$

(Again, note that this is different from  $\overline{\int_{\Gamma} z \, dz}$ .)

(b) We set out this solution in a similar style to Example 2.3.

The standard parametrisation of  $\Gamma$ , the unit circle  $\{z : |z| = 1\}$ , is

$$\gamma(t) = e^{it} \quad (t \in [0, 2\pi]);$$

hence

$$z = e^{it}, \quad dz = ie^{it} \, dt.$$

(i) Here  $f(z) = \bar{z} = e^{-it}$ , and

$$\begin{aligned} \int_{\Gamma} \bar{z} \, dz &= \int_0^{2\pi} e^{-it} \times ie^{it} \, dt \\ &= i \int_0^{2\pi} 1 \, dt \\ &= i[t]_0^{2\pi} \\ &= 2\pi i. \end{aligned}$$

(ii) Here  $f(z) = z^2 = e^{2it}$ , and

$$\begin{aligned} \int_{\Gamma} z^2 \, dz &= \int_0^{2\pi} e^{2it} \times ie^{it} \, dt \\ &= \int_0^{2\pi} ie^{3it} \, dt \\ &= \int_0^{2\pi} i(\cos 3t + i \sin 3t) \, dt \\ &= \int_0^{2\pi} (-\sin 3t) \, dt + i \int_0^{2\pi} \cos 3t \, dt \\ &= \left[ \frac{1}{3} \cos 3t \right]_0^{2\pi} + i \left[ \frac{1}{3} \sin 3t \right]_0^{2\pi} \\ &= 0. \end{aligned}$$

(c) The standard parametrisation of  $\Gamma$ , the upper half of the circle with centre 0 and radius 2, traversed from 2 to  $-2$ , is

$$\gamma(t) = 2e^{it} \quad (t \in [0, \pi]);$$

hence

$$\gamma'(t) = 2ie^{it}.$$

(i) Here  $f(z) = 1/z$ , and

$$\begin{aligned} \int_{\Gamma} \frac{1}{z} \, dz &= \int_0^{\pi} \frac{1}{2e^{it}} \times 2ie^{it} \, dt \\ &= i \int_0^{\pi} 1 \, dt \\ &= i[t]_0^{\pi} \\ &= \pi i. \end{aligned}$$

(ii) Here  $f(z) = |z|$ , and

$$\begin{aligned}\int_{\Gamma} |z| dz &= \int_0^{\pi} |2e^{it}| \times 2ie^{it} dt \\ &= \int_0^{\pi} 4i(\cos t + i \sin t) dt \\ &= \int_0^{\pi} (-4 \sin t) dt + i \int_0^{\pi} 4 \cos t dt \\ &= [4 \cos t]_0^{\pi} + i [4 \sin t]_0^{\pi} \\ &= -8.\end{aligned}$$

## Solution to Exercise 2.7

(a)  $\Gamma = \Gamma_1 + \Gamma_2$ , where  $\Gamma_1$  is the line segment from 0 to  $i$  and  $\Gamma_2$  is the line segment from  $i$  to  $1 + i$ .

We choose to use the standard parametrisations

$$\gamma_1(t) = it \quad (t \in [0, 1]),$$

$$\gamma_2(t) = t + i \quad (t \in [0, 1]).$$

Then  $\gamma_1'(t) = i$ ,  $\gamma_2'(t) = 1$ . Hence

$$\begin{aligned}\int_{\Gamma} \operatorname{Re} z dz &= \int_{\Gamma_1} \operatorname{Re} z dz + \int_{\Gamma_2} \operatorname{Re} z dz \\ &= \int_0^1 \operatorname{Re}(it) \times i dt + \int_0^1 \operatorname{Re}(t + i) \times 1 dt \\ &= \int_0^1 0 dt + \int_0^1 t dt \\ &= \left[\frac{1}{2}t^2\right]_0^1 = \frac{1}{2}.\end{aligned}$$

(b)  $\Gamma = \Gamma_1 + \Gamma_2$ , where  $\Gamma_1$  is the line segment from 0 to 1 and  $\Gamma_2$  is the line segment from 1 to  $1 + i$ .

We choose to use the standard parametrisations

$$\gamma_1(t) = t \quad (t \in [0, 1]),$$

$$\gamma_2(t) = 1 + it \quad (t \in [0, 1]).$$

Then  $\gamma_1'(t) = 1$ ,  $\gamma_2'(t) = i$ . Hence

$$\begin{aligned}\int_{\Gamma} \operatorname{Re} z dz &= \int_{\Gamma_1} \operatorname{Re} z dz + \int_{\Gamma_2} \operatorname{Re} z dz \\ &= \int_0^1 \operatorname{Re} t \times 1 dt + \int_0^1 \operatorname{Re}(1 + it) \times i dt \\ &= \int_0^1 t dt + i \int_0^1 1 dt \\ &= \left[\frac{1}{2}t^2\right]_0^1 + i[t]_0^1 = \frac{1}{2} + i.\end{aligned}$$

(Note that the integrals in parts (a) and (b) have different values.)

## Solution to Exercise 3.1

$$(a) \quad F(z) = \frac{1}{3i} e^{3iz} \quad (z \in \mathbb{C})$$

$$(b) \quad F(z) = i(1 + iz)^{-1} = (z - i)^{-1} \quad (z \in \mathbb{C} - \{i\})$$

$$(c) \quad F(z) = \operatorname{Log} z \quad (\operatorname{Re} z > 0)$$

## Solution to Exercise 3.2

Let  $f(z) = e^{3iz}$ ,  $F(z) = e^{3iz}/(3i)$  and  $\mathcal{R} = \mathbb{C}$ .

Then  $f$  is continuous on  $\mathcal{R}$ , and  $F$  is a primitive of  $f$  on  $\mathcal{R}$ . Thus, by the Fundamental Theorem of Calculus,

$$\begin{aligned}\int_{\Gamma} e^{3iz} dz &= F(-2) - F(2) \\ &= \frac{1}{3i} (e^{-6i} - e^{6i}) = -\frac{2}{3} \sin 6.\end{aligned}$$

The final simplification follows from the formula

$$\sin z = \frac{1}{2i} (e^{iz} - e^{-iz}),$$

with  $z = 6$ .

## Solution to Exercise 3.3

(a) Let  $f(z) = e^{-\pi z}$ ,  $F(z) = -e^{-\pi z}/\pi$  and  $\mathcal{R} = \mathbb{C}$ . Then  $f$  is continuous on  $\mathcal{R}$ , and  $F$  is a primitive of  $f$  on  $\mathcal{R}$ . Thus, by the Fundamental Theorem of Calculus,

$$\begin{aligned}\int_{\Gamma} e^{-\pi z} dz &= F(i) - F(-i) \\ &= (-e^{-\pi i}/\pi) - (-e^{\pi i}/\pi) \\ &= 1/\pi - 1/\pi = 0.\end{aligned}$$

(b) Let  $f(z) = (3z - 1)^2$ ,  $F(z) = \frac{1}{9} (3z - 1)^3$  and  $\mathcal{R} = \mathbb{C}$ . Then  $f$  is continuous on  $\mathcal{R}$ , and  $F$  is a primitive of  $f$  on  $\mathcal{R}$ . Thus, by the Fundamental Theorem of Calculus,

$$\begin{aligned}\int_{\Gamma} (3z - 1)^2 dz &= F(2i + \frac{1}{3}) - F(2) \\ &= \frac{1}{9} (6i)^3 - \frac{1}{9} \times 5^3 \\ &= -\frac{1}{9} (125 + 216i).\end{aligned}$$

(c) Let  $f(z) = \sinh z$ ,  $F(z) = \cosh z$  and  $\mathcal{R} = \mathbb{C}$ . Then  $f$  is continuous on  $\mathcal{R}$ , and  $F$  is a primitive of  $f$  on  $\mathcal{R}$ . Thus, by the Fundamental Theorem of Calculus,

$$\begin{aligned} \int_{\Gamma} \sinh z \, dz &= F(1) - F(i) \\ &= \cosh 1 - \cosh i \\ &= \cosh 1 - \cos 1. \end{aligned}$$

(d) The integrand  $e^{\sin z} \cos z$  can be written as  $\exp(\sin z) \times \sin' z$ ,

which equals  $(\exp \circ \sin)'(z)$ , by the Chain Rule. So let  $f(z) = \exp(\sin z) \cos z$ ,  $F(z) = \exp(\sin z)$  and  $\mathcal{R} = \mathbb{C}$ . Then  $f$  is continuous on  $\mathcal{R}$ , and  $F$  is a primitive of  $f$  on  $\mathcal{R}$ . Thus, by the Fundamental Theorem of Calculus,

$$\begin{aligned} \int_{\Gamma} e^{\sin z} \cos z \, dz &= F(\pi/2) - F(0) \\ &= \exp(\sin(\pi/2)) - \exp(\sin 0) \\ &= e - 1. \end{aligned}$$

*Remark:* If you have a good deal of experience at differentiating and integrating real and complex functions, then you may have chosen to write down the primitive  $F(z) = e^{\sin z}$  of  $f(z) = e^{\sin z} \cos z$  straight away.

(e) The integrand  $\sin z / \cos^2 z$  can be written as

$$-\frac{1}{\cos^2 z} \cos' z,$$

which equals

$$(h \circ \cos)'(z), \quad \text{where } h(z) = 1/z.$$

So let

$$\begin{aligned} f(z) &= \sin z / \cos^2 z, \\ F(z) &= h(\cos z) = 1/\cos z, \\ \mathcal{R} &= \mathbb{C} - \{(n + \frac{1}{2})\pi : n \in \mathbb{Z}\}. \end{aligned}$$

Then  $f$  is continuous on  $\mathcal{R}$ , and  $F$  is a primitive of  $f$  on  $\mathcal{R}$ . Thus, by the Fundamental Theorem of Calculus,

$$\begin{aligned} \int_{\Gamma} \frac{\sin z}{\cos^2 z} \, dz &= F(\pi) - F(0) \\ &= \frac{1}{\cos \pi} - \frac{1}{\cos 0} \\ &= -1 - 1 = -2. \end{aligned}$$

(In this solution, note that the region  $\mathcal{R}$  does not

contain the point  $\pi/2$ , as  $\cos \pi/2 = 0$ ; thus  $\Gamma$  cannot be chosen to be a path that contains  $\pi/2$ . In particular, the real integral  $\int_0^{\pi} \frac{\sin x}{\cos^2 x} \, dx$  does not exist.)

### Solution to Exercise 3.4

(a) We take  $f(z) = z$ ,  $g(z) = \sinh z$  and  $\mathcal{R} = \mathbb{C}$ .

Then  $f$  and  $g$  are analytic on  $\mathcal{R}$ , and  $f'(z) = 1$  and  $g'(z) = \cosh z$  are continuous on  $\mathcal{R}$ .

Integrating by parts, we obtain

$$\begin{aligned} \int_{\Gamma} z \cosh z \, dz &= [z \sinh z]_0^{\pi i} - \int_{\Gamma} 1 \times \sinh z \, dz \\ &= (\pi i \sinh \pi i - 0) - [\cosh z]_0^{\pi i} \\ &= \pi i \times i \sin \pi - (\cos \pi - \cosh 0) \\ &= 0 - (-1 - 1) = 2. \end{aligned}$$

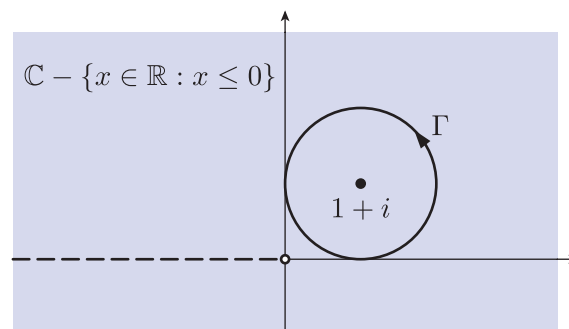
(b) We take  $f(z) = \text{Log } z$ ,  $g(z) = z$  and  $\mathcal{R} = \mathbb{C} - \{x \in \mathbb{R} : x \leq 0\}$ . Then  $f$  and  $g$  are analytic on  $\mathcal{R}$ , and  $f'(z) = 1/z$  and  $g'(z) = 1$  are continuous on  $\mathcal{R}$ .

Integrating by parts, we obtain

$$\begin{aligned} \int_{\Gamma} \text{Log } z \, dz &= [z \text{Log } z]_1^i - \int_{\Gamma} \frac{1}{z} \times z \, dz \\ &= i \text{Log } i - \text{Log } 1 - [z]_1^i \\ &= -\pi/2 - (i - 1) = (1 - \pi/2) - i. \end{aligned}$$

### Solution to Exercise 3.5

(a) Let  $f(z) = 1/z$ ,  $F(z) = \text{Log } z$  and  $\mathcal{R} = \mathbb{C} - \{x \in \mathbb{R} : x \leq 0\}$ . Then  $f$  is continuous on  $\mathcal{R}$ ,  $F$  is a primitive of  $f$  on  $\mathcal{R}$ , and  $\Gamma$  lies in  $\mathcal{R}$ , as illustrated in the following figure.



Thus, by the Closed Contour Theorem,

$$\int_{\Gamma} 1/z \, dz = 0.$$

(b) Let  $f(z) = 1/z^2$ ,  $F(z) = -1/z$  and  $\mathcal{R} = \mathbb{C} - \{0\}$ . Then  $f$  is continuous on  $\mathcal{R}$ ,  $F$  is a primitive of  $f$  on  $\mathcal{R}$ , and  $\Gamma$  lies in  $\mathcal{R}$ . Thus, by the Closed Contour Theorem,

$$\int_{\Gamma} 1/z^2 dz = 0.$$

### Solution to Exercise 3.6

Let  $F(z) = F_1(z) - F_2(z)$ . Since  $F_1$  and  $F_2$  are both primitives of  $f$  on  $\mathcal{R}$ , we see that

$$F_1'(z) = F_2'(z) = f(z), \quad \text{for all } z \text{ in } \mathcal{R}.$$

Hence  $F$  is analytic on  $\mathcal{R}$ , and

$$F'(z) = F_1'(z) - F_2'(z) = 0, \quad \text{for all } z \text{ in } \mathcal{R}.$$

It follows from the Zero Derivative Theorem (with  $F$  in place of  $f$ ) that

$$F(z) = c, \quad \text{for all } z \text{ in } \mathcal{R},$$

where  $c$  is a complex constant.

Thus  $F_1(z) = F_2(z) + c$ , for all  $z$  in  $\mathcal{R}$ .

### Solution to Exercise 3.7

In each case,  $f$  is continuous on  $\mathbb{C}$  and has a primitive on  $\mathbb{C}$ , so we can apply the Fundamental Theorem of Calculus to evaluate the integral using any contour  $\Gamma$  from  $-i$  to  $i$ .

$$(a) \int_{\Gamma} 1 dz = [z]_{-i}^i = i - (-i) = 2i$$

$$(b) \int_{\Gamma} z dz = [\frac{1}{2}z^2]_{-i}^i = \frac{1}{2}i^2 - \frac{1}{2}(-i)^2 = 0$$

$$(c) \int_{\Gamma} (5z^4 + 3iz^2) dz = [z^5 + iz^3]_{-i}^i \\ = (i + 1) - (-i - 1) \\ = 2 + 2i$$

$$(d) \int_{\Gamma} (1 + 2iz)^9 dz = [(1 + 2iz)^{10}/(10 \times 2i)]_{-i}^i \\ = ((-1)^{10} - 3^{10})/(20i) \\ = \frac{3^{10} - 1}{20}i$$

$$(e) \int_{\Gamma} e^{-iz} dz = [e^{-iz}/(-i)]_{-i}^i \\ = (e - e^{-1})/(-i) = 2i \sinh 1$$

$$(f) \int_{\Gamma} \sin z dz = [-\cos z]_{-i}^i \\ = -\cos i + \cos(-i) = 0$$

$$(g) \text{ A primitive of } f(z) = ze^{z^2} \text{ is} \\ F(z) = \frac{1}{2}e^{z^2}.$$

Hence

$$\int_{\Gamma} ze^{z^2} dz = [\frac{1}{2}e^{z^2}]_{-i}^i \\ = \frac{1}{2}(e^{-1} - e^{-1}) = 0.$$

$$(h) \text{ A primitive of } f(z) = z^3 \cosh(z^4) \text{ is} \\ F(z) = \frac{1}{4} \sinh(z^4).$$

Hence

$$\int_{\Gamma} z^3 \cosh(z^4) dz = [\frac{1}{4} \sinh(z^4)]_{-i}^i \\ = \frac{1}{4}(\sinh 1 - \sinh 1) = 0.$$

(i) Let  $g(z) = z$ ,  $h(z) = e^z$ . Then  $g$  and  $h$  are entire (that is,  $g$  and  $h$  are differentiable on the whole of  $\mathbb{C}$ ), and  $g'$  and  $h'$  are entire and hence continuous. Then, using Integration by Parts (Theorem 3.3), we have

$$\int_{\Gamma} ze^z dz = [ze^z]_{-i}^i - \int_{\Gamma} 1 \times e^z dz \\ = (ie^i - (-i)e^{-i}) - \int_{\Gamma} e^z dz \\ = i(e^i + e^{-i}) - [e^z]_{-i}^i \\ = 2i \cos 1 - (e^i - e^{-i}) \\ = 2i \cos 1 - 2i \sin 1 \\ = 2(\cos 1 - \sin 1)i.$$

### Solution to Exercise 3.8

(a) Let  $f(z) = 1/z$ ,  $F(z) = \text{Log } z$  and  $\mathcal{R} = \mathbb{C} - \{x \in \mathbb{R} : x \leq 0\}$ . Then  $f$  is continuous on  $\mathcal{R}$ ,  $F$  is a primitive of  $f$  on  $\mathcal{R}$ , and  $\Gamma$  is a contour in  $\mathcal{R}$ . Thus, by the Fundamental Theorem of Calculus,

$$\int_{\Gamma} \frac{1}{z} dz = [\text{Log } z]_{-i}^i \\ = \text{Log } i - \text{Log }(-i) \\ = \frac{\pi}{2}i - \left(-\frac{\pi}{2}i\right) = \pi i.$$

(b) Let  $f(z) = \sqrt{z}$ ,  $F(z) = \frac{2}{3}z^{3/2}$  and  $\mathcal{R} = \mathbb{C} - \{x \in \mathbb{R} : x \leq 0\}$ . Then  $f$  is continuous on  $\mathcal{R}$ ,  $F$  is a primitive of  $f$  on  $\mathcal{R}$ , and  $\Gamma$  is a contour in  $\mathcal{R}$ . Thus, by the Fundamental Theorem of Calculus,

$$\begin{aligned} \int_{\Gamma} \sqrt{z} dz &= \left[ \frac{2}{3} z^{3/2} \right]_{-i}^i \\ &= \frac{2}{3} \left( i^{3/2} - (-i)^{3/2} \right) \\ &= \frac{2}{3} \left( \exp\left(\frac{3}{2} \operatorname{Log} i\right) - \exp\left(\frac{3}{2} \operatorname{Log}(-i)\right) \right) \\ &= \frac{2}{3} \left( \exp\left(\frac{3\pi i}{4}\right) - \exp\left(-\frac{3\pi i}{4}\right) \right) \\ &= \frac{2}{3} \left( 2i \sin \frac{3\pi}{4} \right) \\ &= \frac{2\sqrt{2}}{3} i. \end{aligned}$$

(c) The function

$$f(z) = \sin^2 z = \frac{1}{2}(1 - \cos 2z)$$

is continuous and has an entire primitive  $F(z) = \frac{1}{2}(z - \frac{1}{2} \sin 2z)$ . Thus, by the Closed Contour Theorem,

$$\int_{\Gamma} \sin^2 z dz = 0.$$

(d) Let  $f(z) = 1/z^3$ ,  $F(z) = -1/(2z^2)$  and  $\mathcal{R} = \mathbb{C} - \{0\}$ . Then  $f$  is continuous on  $\mathcal{R}$ ,  $F$  is a primitive of  $f$  on  $\mathcal{R}$ , and  $\Gamma$  is a contour in  $\mathcal{R}$ . Thus, by the Closed Contour Theorem,

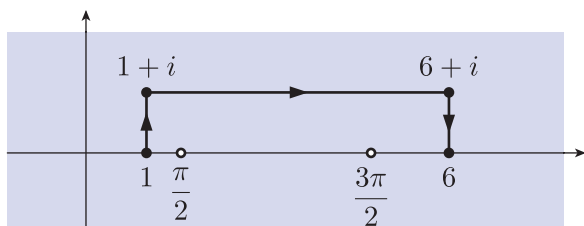
$$\int_{\Gamma} \frac{1}{z^3} dz = 0.$$

### Solution to Exercise 3.9

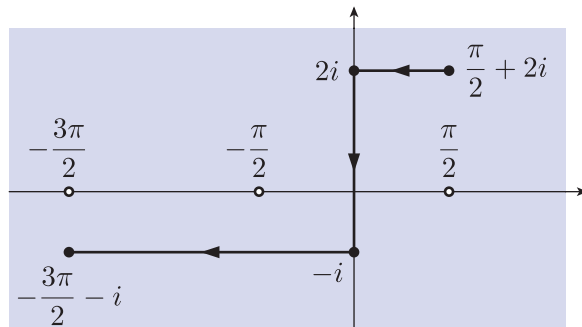
The domain of  $\tan$  is the region

$$\mathcal{R} = \mathbb{C} - \left\{ \left(n + \frac{1}{2}\right)\pi : n \in \mathbb{Z} \right\}.$$

(a) The figure shows one grid path in  $\mathcal{R}$  from 1 to 6 (there are many others).



(b) The figure shows one grid path in  $\mathcal{R}$  from  $\frac{\pi}{2} + 2i$  to  $-\frac{3\pi}{2} - i$  (again, there are many others).



### Solution to Exercise 4.1

(a) The standard parametrisation of the circle with centre  $\alpha$  and radius  $r$  is

$$\gamma(t) = \alpha + re^{it} \quad (t \in [0, 2\pi]).$$

Then  $\gamma'(t) = rie^{it}$ , so

$$|\gamma'(t)| = r.$$

Thus the required length is

$$\int_0^{2\pi} r dt = 2\pi r.$$

(b) Since  $\gamma(t) = t + i \cosh t$  ( $t \in [0, 1]$ ), we have

$$\gamma'(t) = 1 + i \sinh t,$$

and, since  $\cosh t > 0$ ,

$$|\gamma'(t)| = \sqrt{1 + \sinh^2 t} = \cosh t.$$

Thus the required length is

$$\int_0^1 \cosh t dt = [\sinh t]_0^1 = \sinh 1.$$

### Solution to Exercise 4.2

(a) The length  $L$  of  $\Gamma$  is  $10\pi$ , as  $\Gamma$  is a circle of radius 5. To find a value for  $M$ , observe that for  $z \in \Gamma$  we have

$$|z^2| = 25 \quad \text{and} \quad |e^{3z}| = e^{3\operatorname{Re} z} \leq e^{15},$$

since  $\operatorname{Re} z \leq 5$ , for  $|z| = 5$ . Thus

$$\left| \frac{e^{3z}}{z^2} \right| = \frac{|e^{3z}|}{|z^2|} \leq \frac{e^{15}}{25}, \quad \text{for } z \in \Gamma,$$

so we can take  $M = e^{15}/25$ .

Since  $f(z) = e^{3z}/z^2$  is continuous on  $\mathbb{C} - \{0\}$ , it is also continuous on  $\Gamma$ , so we can apply the Estimation Theorem to obtain

$$\left| \int_{\Gamma} \frac{e^{3z}}{z^2} dz \right| \leq \frac{e^{15}}{25} \times 10\pi = \frac{2\pi}{5} e^{15}.$$

(b) The length  $L$  of  $\Gamma$  is 40, as  $\Gamma$  is a square of side 10. To find a value for  $M$ , observe that for  $z \in \Gamma$  we have

$$|e^{3z}| = e^{3\operatorname{Re} z} \leq e^{15},$$

since  $\operatorname{Re} z \leq 5$ , for  $z \in \Gamma$ . Also, since  $\Gamma$  lies in the set  $\{z : |z| \geq 5\}$ , we see that

$$|z^2| = |z|^2 \geq 25, \quad \text{for } z \in \Gamma.$$

Thus

$$\left| \frac{e^{3z}}{z^2} \right| = \frac{|e^{3z}|}{|z^2|} \leq \frac{e^{15}}{25}, \quad \text{for } z \in \Gamma,$$

so we can take  $M = e^{15}/25$ . Since  $f(z) = e^{3z}/z^2$  is continuous on  $\Gamma$ , we can apply the Estimation Theorem to obtain

$$\left| \int_{\Gamma} \frac{e^{3z}}{z^2} dz \right| \leq \frac{e^{15}}{25} \times 40 = \frac{8}{5} e^{15}.$$

### Solution to Exercise 4.3

We need to find a value for  $L$ , the length of the contour  $\Gamma$ , and an upper estimate  $M$  for  $|f(z)|$  on  $\Gamma$ , where

$$f(z) = \frac{3z - 4}{2z - 5}.$$

We have  $L = 6\pi$ , as  $\Gamma$  is a circle of radius 3. To find a value for  $M$ , we need an *upper* estimate for  $|3z - 4|$  and a *lower* estimate for  $|2z - 5|$  on  $\Gamma$ . By the Triangle Inequality (usual form), we have

$$\begin{aligned} |3z - 4| &\leq |3z| + |-4| \\ &= 9 + 4 = 13, \quad \text{for } z \in \Gamma, \end{aligned}$$

and by the Triangle Inequality (backwards form), we have

$$\begin{aligned} |2z - 5| &\geq ||2z| - 5| \\ &= 6 - 5 = 1, \quad \text{for } z \in \Gamma. \end{aligned}$$

Thus

$$\left| \frac{3z - 4}{2z - 5} \right| \leq \frac{13}{1} = 13, \quad \text{for } z \in \Gamma,$$

so we can take  $M = 13$ . Since  $f$  is continuous on  $\mathbb{C} - \{5/2\}$ , it is also continuous on  $\Gamma$ , so we can apply the Estimation Theorem to obtain

$$\left| \int_{\Gamma} \frac{3z - 4}{2z - 5} dz \right| \leq 13 \times 6\pi = 78\pi.$$

### Solution to Exercise 4.4

The length  $L$  of  $\Gamma$  is  $4\pi$ , as  $\Gamma$  is a semicircle of radius 4. To find a value for  $M$ , note that, by the Triangle Inequality (backwards form), we have

$$\begin{aligned} |z^2 - 9| &\geq ||z^2| - 9| \\ &= |16 - 9| = 7, \quad \text{for } z \in \Gamma. \end{aligned}$$

For the exponential term, we can write

$$\begin{aligned} |e^{2iz}| &= |e^{2i(x+iy)}| \\ &= |e^{2ix}| |e^{-2y}| \\ &= |e^{-2y}| \\ &= e^{-2y} \\ &\leq e^0 = 1, \quad \text{for } z \in \Gamma, \end{aligned}$$

since  $y \geq 0$ , for  $z \in \Gamma$ . Thus

$$\left| \frac{e^{2iz}}{z^2 - 9} \right| \leq \frac{1}{7}, \quad \text{for } z \in \Gamma,$$

so we can take  $M = 1/7$ . Since  $f(z) = e^{2iz}/(z^2 - 9)$  is continuous on  $\mathbb{C} - \{3, -3\}$  (by the Composition Rule and Quotient Rule for continuous functions), it is also continuous on  $\Gamma$ , so we can apply the Estimation Theorem to obtain

$$\left| \int_{\Gamma} \frac{e^{2iz}}{z^2 - 9} dz \right| \leq \frac{1}{7} \times 4\pi = \frac{4\pi}{7}.$$

### Solution to Exercise 4.5

In each case we check that the hypotheses of the Estimation Theorem apply; that is, we check that  $f$  is continuous on  $\Gamma$  and  $|f(z)| \leq M$ , for  $z \in \Gamma$  and some positive number  $M$ . Observe that  $\Gamma$  has length  $L = 6\pi$ .

(a) The function  $f(z) = (z - 3)/(z + 2)$  is continuous on  $\mathbb{C} - \{-2\}$ , so it is continuous on  $\Gamma$ .

To find a value for  $M$ , note that, by the Triangle Inequality (usual form), we have

$$\begin{aligned} |z - 3| &\leq |z| + |-3| \\ &= 3 + 3 = 6, \quad \text{for } z \in \Gamma, \end{aligned}$$

and, by the Triangle Inequality (backwards form), we have

$$\begin{aligned} |z + 2| &\geq ||z| - |2|| \\ &= 3 - 2 = 1, \quad \text{for } z \in \Gamma. \end{aligned}$$

Thus

$$\left| \frac{z-3}{z+2} \right| \leq \frac{6}{1} = 6, \quad \text{for } z \in \Gamma,$$

so we can take  $M = 6$ . It follows from the Estimation Theorem that

$$\left| \int_{\Gamma} \frac{z-3}{z+2} dz \right| \leq 6 \times 6\pi = 36\pi.$$

**(b)** The function  $f(z) = (z^2 + 4)/(z^2 - 4)$  is continuous on  $\mathbb{C} - \{2, -2\}$ , so it is continuous on  $\Gamma$ .

To find a value for  $M$ , note that, by the Triangle Inequality (usual form), we have

$$\begin{aligned} |z^2 + 4| &\leq |z^2| + |4| \\ &= 9 + 4 = 13, \quad \text{for } z \in \Gamma, \end{aligned}$$

and, by the Triangle Inequality (backwards form), we have

$$\begin{aligned} |z^2 - 4| &\geq ||z^2| - |-4|| \\ &= 9 - 4 = 5, \quad \text{for } z \in \Gamma. \end{aligned}$$

Thus

$$\left| \frac{z^2 + 4}{z^2 - 4} \right| \leq \frac{13}{5}, \quad \text{for } z \in \Gamma,$$

so we can take  $M = 13/5$ . It follows from the Estimation Theorem that

$$\left| \int_{\Gamma} \frac{z^2 + 4}{z^2 - 4} dz \right| \leq \frac{13}{5} \times 6\pi = \frac{78\pi}{5}.$$

**(c)** The function  $f(z) = \sin z/(1 + z^2)$  is continuous on  $\mathbb{C} - \{i, -i\}$ , so it is continuous on  $\Gamma$ .

To find a value for  $M$ , note that, by the Triangle Inequality (usual form), we have

$$\begin{aligned} |\sin z| &= \frac{1}{2} |e^{iz} - e^{-iz}| \\ &\leq \frac{1}{2} (|e^{iz}| + |e^{-iz}|) \\ &= \frac{1}{2} (e^{\operatorname{Re}(iz)} + e^{\operatorname{Re}(-iz)}) \\ &= \frac{1}{2} (e^{-y} + e^y) \quad (\text{where } z = x + iy) \\ &\leq \frac{1}{2} (e^3 + e^3) = e^3, \quad \text{for } z \in \Gamma, \end{aligned}$$

and, by the Triangle Inequality (backwards form), we have

$$\begin{aligned} |1 + z^2| &\geq |1 - |z^2|| \\ &= |1 - 9| = 8, \quad \text{for } z \in \Gamma. \end{aligned}$$

Thus

$$\left| \frac{\sin z}{1 + z^2} \right| \leq \frac{e^3}{8}, \quad \text{for } z \in \Gamma,$$

so we can take  $M = e^3/8$ . It follows from the Estimation Theorem that

$$\left| \int_{\Gamma} \frac{\sin z}{1 + z^2} dz \right| \leq \frac{e^3}{8} \times 6\pi = \frac{3e^3\pi}{4}.$$



Unit B2

Cauchy's Theorem



# Introduction

In this unit we present three of the most important results in complex analysis: Cauchy's Theorem, Cauchy's Integral Formula and Cauchy's  $n$ th Derivative Formula. These results lie at the heart of complex analysis and give the subject much of its distinctive flavour.

In the previous unit we proved the Closed Contour Theorem (Theorem 3.4 of Unit B1), which states that if  $f$  is a function that is continuous on a region  $\mathcal{R}$  and has a primitive on  $\mathcal{R}$ , then

$$\int_{\Gamma} f(z) dz = 0,$$

for any closed contour  $\Gamma$  in  $\mathcal{R}$ .

Cauchy's Theorem has the same conclusion as the Closed Contour Theorem. It states that if  $\mathcal{R}$  is a region of a particular type and if  $f$  is analytic on  $\mathcal{R}$ , then

$$\int_{\Gamma} f(z) dz = 0,$$

for any closed contour  $\Gamma$  in  $\mathcal{R}$ . Cauchy's Theorem is discussed in Section 1, where we give an outline proof. The full details of this proof, which are rather involved, appear in Section 5.

Section 2 is devoted to a discussion of Cauchy's remarkable Integral Formula, which expresses the value of an analytic function at any point *inside* a contour in terms of the values of the function *on* the contour.

Using the Integral Formula, we prove a spectacular result, known as Liouville's Theorem, which states that any function that is entire (differentiable on  $\mathbb{C}$ ) and bounded must be constant. This is a result that has no analogue in real analysis, where there are many non-constant functions, such as  $x \mapsto \sin x$  and  $x \mapsto e^{-x^2}$ , that are both differentiable on  $\mathbb{R}$  and bounded.

We use Liouville's Theorem to prove the Fundamental Theorem of Algebra, which says that every non-constant polynomial function has at least one zero. This is another result that has no analogue in real analysis, because there are non-constant real polynomial functions such as  $x \mapsto x^2 + 1$  that never take the value zero.

In Section 3 we consider a further result due to Cauchy, which gives a formula (in terms of an integral) for the derivative of an analytic function. Once again there are many surprises: not only do we use integrals when we want to differentiate, but we find that if a function can be differentiated once on a region, then it can be differentiated as many times as we like – again, a result that has no analogue in real analysis.

After this wealth of unexpected results and new ideas, you may feel the need for a change of pace. This is provided by Section 4, a revision section, in which we consider examples of integrals that can be evaluated by a variety of methods.

## Unit guide

Most of the material in this unit is essential for your later work, and you should make sure that you become familiar with it. In particular, you will need to understand the results of Sections 1, 2 and 3, and you should know how to use them. However, if you are short of time, then you can omit Section 5 on a first reading.

The revision section (Section 4) contains no new material, as it reviews techniques from this unit and the previous one.

## 1 Cauchy's Theorem

After working through this section, you should be able to:

- explain what is meant by a *simple-closed path* and a *simply connected region*
- state the Jordan Curve Theorem
- state and use Cauchy's Theorem
- state the Primitive Theorem and explain its role in the proof of Cauchy's Theorem
- state and use the Contour Independence Theorem and the Shrinking Contour Theorem.

### 1.1 Simply connected regions

In this subsection we introduce the type of region that we will need in the statement of Cauchy's Theorem.

The Closed Contour Theorem tells us that if  $f$  is a function that is continuous on a region  $\mathcal{R}$  and has a primitive on  $\mathcal{R}$ , then

$$\int_{\Gamma} f(z) dz = 0, \quad (1.1)$$

for any closed contour  $\Gamma$  in  $\mathcal{R}$ . It is tempting to conjecture that the same conclusion will hold if we assume that  $f$  is analytic on the region  $\mathcal{R}$ .

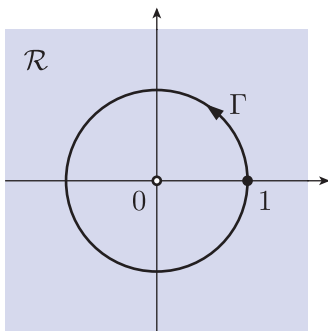
Consider, however, Example 2.3 of Unit B1, which says that

$$\int_{\Gamma} \frac{1}{z} dz = 2\pi i,$$

where  $\Gamma$  is the unit circle  $\{z : |z| = 1\}$ . Here the integrand  $f(z) = 1/z$  is analytic on the region  $\mathcal{R} = \mathbb{C} - \{0\}$ , which contains  $\Gamma$  (see Figure 1.1), and yet

$$\int_{\Gamma} f(z) dz \neq 0.$$

In this example, however, the closed contour  $\Gamma$  encloses the point 0, so the inside of  $\Gamma$  does not belong to the region  $\mathcal{R}$ . Cauchy realised that the key to proving equation (1.1) for an analytic function  $f$  on a region  $\mathcal{R}$  is to insist that the closed contour  $\Gamma$  and its inside lie in  $\mathcal{R}$ .

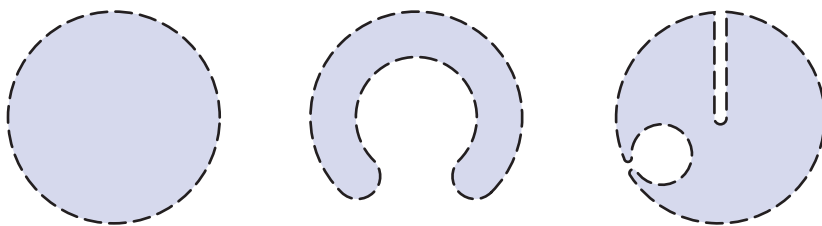


**Figure 1.1** The point 0 inside the unit circle  $\Gamma$

This idea leads us to make the following informal definition.

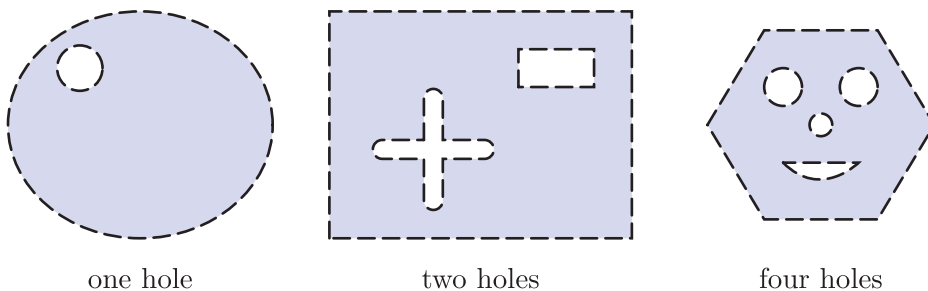
A region  $\mathcal{R}$  is **simply connected** if it has no holes in it.

For example, the regions in Figure 1.2 are simply connected, as they have no holes in them.



**Figure 1.2** Simply connected regions

However, the regions in Figure 1.3 are not simply connected, as each has at least one hole in it.



**Figure 1.3** Regions that are not simply connected

### Exercise 1.1

Determine which of the following regions are simply connected.

- (a)

(b)

(c)

(d)
- (e)  $\{z : -1 < \operatorname{Im} z < 1\}$       (f)  $\{z : |z| > 3\}$       (g)  $\mathbb{C}$   
 (h)  $\{z : -\pi < \operatorname{Arg} z < \pi, 1 < |z| < 2\}$

In order to define a simply connected region formally, we need the concept of a *simple-closed path*; this is a closed path that does not intersect itself (except at the initial and final point). We also define the notion of a *simple path*, which is a path that does not intersect itself.

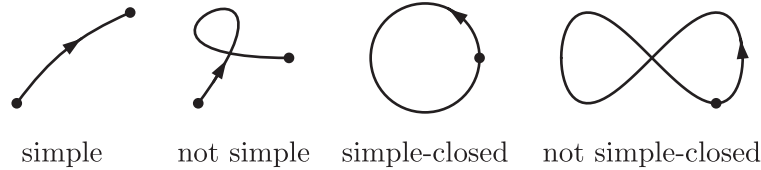
### Definitions

A path  $\Gamma : \gamma(t)$  ( $t \in [a, b]$ ) is **simple** if  $\gamma$  is one-to-one on  $[a, b]$ .

A path  $\Gamma : \gamma(t)$  ( $t \in [a, b]$ ) is **simple-closed** if it is closed *and*  $\gamma$  is one-to-one on  $[a, b]$ .

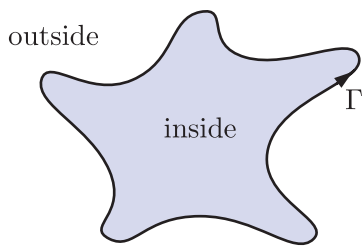
Since a contour is a special type of path, we also speak of **simple contours** and **simple-closed contours**.

Figure 1.4 illustrates these definitions – on the closed paths the dot denotes the point  $\gamma(a) = \gamma(b)$ .



**Figure 1.4** Paths of different types

A circle (which is a simple-closed path) has the property that it divides the complex plane into two regions: the inside of the circle and the outside of the circle. It seems clear that every simple-closed path must have an inside and an outside, but this is surprisingly difficult to prove (because some simple-closed paths are complicated). We therefore only state but do not prove this general result, which is known as the Jordan Curve Theorem.



**Figure 1.5** The inside and outside of a simple-closed path  $\Gamma$

### Theorem 1.1 Jordan Curve Theorem

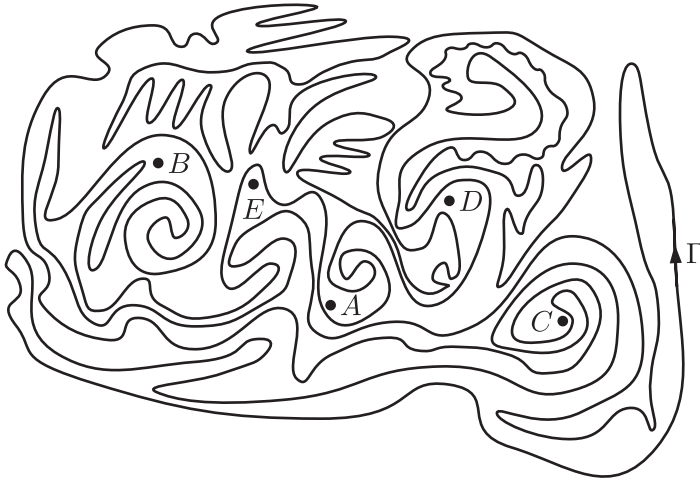
The complement  $\mathbb{C} - \Gamma$  of a simple-closed path  $\Gamma$  is the union of two disjoint regions, one bounded and the other unbounded.

The bounded region is called the **inside** of  $\Gamma$  and the unbounded region is called the **outside** of  $\Gamma$ ; see Figure 1.5.

The following exercise will give you some idea of the difficulty of identifying the inside and outside of a complicated simple-closed path. For many simple-closed paths, however, it is clear which is the inside and which is the outside.

## Exercise 1.2

- (a) By shading the inside of the following simple-closed path  $\Gamma$ , or otherwise, determine which of the points  $A, B, C, D, E$  lie inside  $\Gamma$ .



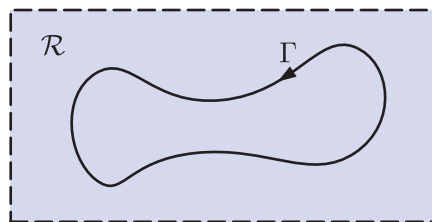
- (b) Try to devise an algorithm for deciding whether a given point  $\alpha$  lies inside  $\Gamma$ . (Don't spend too long on this.)

We can now refine the informal definition of a simply connected region given earlier.

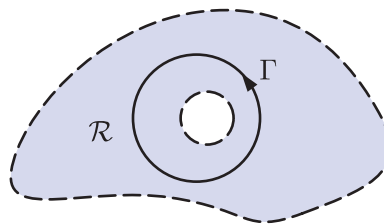
## Definition

A region  $\mathcal{R}$  is **simply connected** if, whenever  $\Gamma$  is a simple-closed path lying in  $\mathcal{R}$ , the inside of  $\Gamma$  also lies in  $\mathcal{R}$ .

For example, the region  $\mathcal{R}$  in Figure 1.6(a) is simply connected because if  $\Gamma$  is *any* simple-closed path in  $\mathcal{R}$ , then the inside of  $\Gamma$  lies completely in  $\mathcal{R}$ . However, the region  $\mathcal{R}$  in Figure 1.6(b) is not simply connected because if  $\Gamma$  is the path illustrated, then the inside of  $\Gamma$  does not lie in  $\mathcal{R}$ .



(a)



(b)

**Figure 1.6** Regions: (a) simply connected, (b) not simply connected

It is difficult to verify that a region is simply connected using this definition, because we need to check that the inside of *every* simple-closed path lies in the region. Instead we generally use the earlier informal definition, and identify a simply connected region as a region without any holes in it.



Camille Jordan

### Origin of the Jordan Curve Theorem

The Jordan Curve Theorem was first proved by the French mathematician Camille Jordan (1838–1922) in his influential text *Cours d'Analyse* (1887). Jordan saw that many of the theorems of complex analysis could be made rigorous only with a precise definition of the inside and outside of a simple-closed path. The proof of his theorem, however, was criticised by others, most notably the American mathematician Oswald Veblen (1880–1960), who published an alternative proof in 1905. Veblen wrote of Jordan's work:

His proof, however, is unsatisfactory to many mathematicians. It assumes the theorem without proof in the important special case of a simple polygon, and of the argument from that point on, one must admit at least that all details are not given.

(Veblen, 1905, p. 83)

Nonetheless, mathematicians of the present day consider Jordan's proof to be acceptable, if presented somewhat obscurely.

The Jordan Curve Theorem was strengthened by the German mathematician Arthur Moritz Schoenflies (1853–1928) in 1906, who proved that the inside of a simple-closed path is simply connected.

## 1.2 Statement of Cauchy's Theorem

Now that we have defined simply connected regions, we can state a central result of complex analysis – Cauchy's Theorem.

### Theorem 1.2 Cauchy's Theorem

Let  $\mathcal{R}$  be a simply connected region, and let  $f$  be a function that is analytic on  $\mathcal{R}$ . Then

$$\int_{\Gamma} f(z) dz = 0,$$

for any closed contour  $\Gamma$  in  $\mathcal{R}$ .

You should be aware that there are other theorems called 'Cauchy's Theorem' in other branches of mathematics (such as group theory), which are quite different to this one.

As an example of Cauchy's Theorem, if we take  $\mathcal{R}$  to be  $\mathbb{C}$ ,  $\Gamma$  to be the unit circle  $C = \{z : |z| = 1\}$ , and  $f(z) = z^2$ , then the conditions of Cauchy's Theorem are satisfied. We deduce that

$$\int_C z^2 dz = 0,$$

a result that also follows from the Closed Contour Theorem.

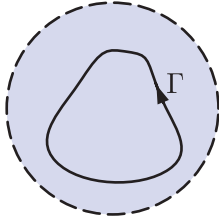
However, Cauchy's Theorem cannot be used to evaluate

$$\int_C \frac{1}{z} dz,$$

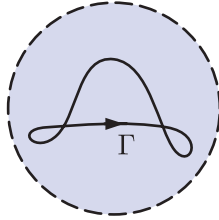
since there is no simply connected region  $\mathcal{R}$  that contains  $C$  and on which the function  $f(z) = 1/z$  is analytic.

### Exercise 1.3

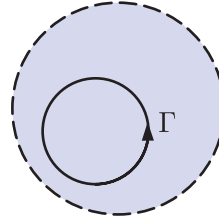
Determine whether the conditions of Cauchy's Theorem are satisfied for each of the following functions  $f$  and contours  $\Gamma$  in the open unit disc.



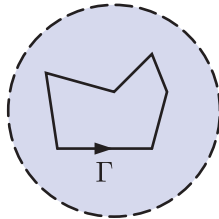
(a)  $f(z) = 1/z$



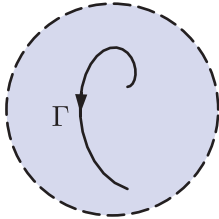
(b)  $f(z) = e^z$



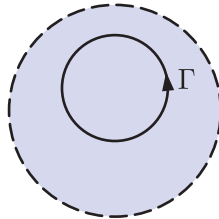
(c)  $f(z) = \text{Log } z$



(d)  $f(z) = 1/(z - 3)$



(e)  $f(z) = z^2$



(f)  $f(z) = |z|$

We now give an example to show how Cauchy's Theorem can be applied in practice.

### Example 1.1

Let  $\Gamma$  be the unit circle  $\{z : |z| = 1\}$ . Prove that

$$\int_{\Gamma} \frac{1}{z+2} dz = 0.$$

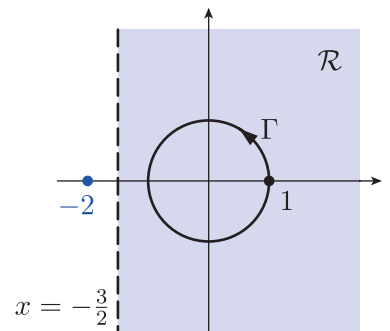
### Solution

We first choose a simply connected region  $\mathcal{R}$  containing  $\Gamma$  on which the function  $f(z) = 1/(z+2)$  is analytic. An example of such a region is

$$\mathcal{R} = \{z : \text{Re } z > -3/2\},$$

since  $\mathcal{R}$  does not contain the point  $-2$  (see Figure 1.7). Then the conditions of Cauchy's Theorem are satisfied, so

$$\int_{\Gamma} \frac{1}{z+2} dz = 0.$$



**Figure 1.7** The unit circle  $\Gamma$  inside  $\mathcal{R}$

## Exercise 1.4

Prove that if  $\Gamma$  is any circle and  $\alpha$  is any point lying *outside*  $\Gamma$ , then

$$\int_{\Gamma} \frac{1}{z - \alpha} dz = 0.$$

We conclude this subsection with an outline of a proof of Cauchy's Theorem. Full details are given in Section 5.

## Outline of a proof of Cauchy's Theorem

Let  $\mathcal{R}$  be a simply connected region, and let  $f$  be a function that is analytic on  $\mathcal{R}$ . We wish to prove that

$$\int_{\Gamma} f(z) dz = 0,$$

for any closed contour  $\Gamma$  in  $\mathcal{R}$ .

The proof is in three stages (the first two of which are special cases of Cauchy's Theorem). The aim is to show that the hypotheses of Cauchy's Theorem imply the hypotheses of the Closed Contour Theorem, and we then apply that theorem to deduce Cauchy's Theorem.

## 1. Cauchy's Theorem for rectangular contours

The first stage is to prove Cauchy's Theorem when  $\Gamma$  is a rectangular contour in  $\mathcal{R}$ , such as the one illustrated in Figure 1.8.

## 2. Cauchy's Theorem for closed grid paths

The second stage is to prove Cauchy's Theorem when  $\Gamma$  is a closed grid path in  $\mathcal{R}$ , such as the one illustrated in Figure 1.9. (The definition of a grid path was given in Subsection 3.2 of Unit B1.)

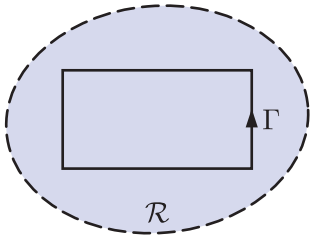
## 3. Primitive Theorem

The final stage is to prove the Primitive Theorem, which says that if  $f$  is a function that is analytic on a simply connected region  $\mathcal{R}$ , then  $f$  has a primitive on  $\mathcal{R}$ . The primitive is constructed using grid paths, and Cauchy's Theorem for closed grid paths is applied to show that this construction does not depend on the choice of grid paths.

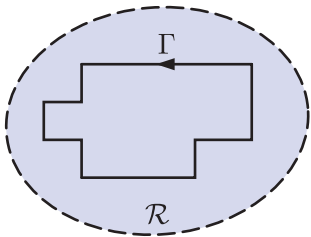
Cauchy's Theorem then follows from the Primitive Theorem and the Closed Contour Theorem.

## 1.3 Some consequences of Cauchy's Theorem

We now use Cauchy's Theorem to deduce some simple, but extremely useful, results. The first of these is a variant of the Contour Independence Theorem (Theorem 3.2 of Unit B1).



**Figure 1.8** A rectangular contour in  $\mathcal{R}$



**Figure 1.9** A closed grid path in  $\mathcal{R}$

**Theorem 1.3 Contour Independence Theorem**

Let  $\mathcal{R}$  be a simply connected region, let  $f$  be a function that is analytic on  $\mathcal{R}$ , and let  $\Gamma_1$  and  $\Gamma_2$  be contours in  $\mathcal{R}$  with the same initial point  $\alpha$  and the same final point  $\beta$ . Then

$$\int_{\Gamma_1} f(z) dz = \int_{\Gamma_2} f(z) dz.$$

Two such contours  $\Gamma_1$  and  $\Gamma_2$  are illustrated in Figure 1.10.

**Exercise 1.5**

By applying Cauchy's Theorem to the closed contour  $\Gamma_1 + \tilde{\Gamma}_2$ , where  $\tilde{\Gamma}_2$  is the *reverse* of  $\Gamma_2$ , prove Theorem 1.3.

Before stating our next result, we need to establish a convention about integrals around simple-closed contours. Recall that a closed path  $\Gamma : \gamma(t)$  ( $t \in [a, b]$ ) is simple-closed if  $\gamma$  is one-to-one on  $[a, b]$ . Thus as  $t$  increases from  $a$  to  $b$ , the point  $\gamma(t)$  traverses  $\Gamma$  exactly once, in either the clockwise or the anticlockwise direction.

**Convention**

Unless otherwise specified, any simple-closed contour  $\Gamma$  appearing in a contour integral will be assumed to be traversed once anticlockwise, with the inside of  $\Gamma$  on the left.

Contours that are traversed in the manner described by the convention are sometimes said to be *positively orientated*.

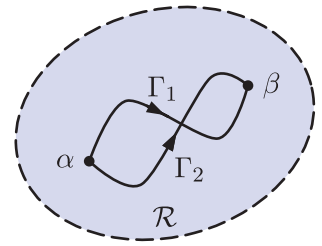
The next result, which will be needed in Section 2 and elsewhere, shows that, under suitable conditions, we can replace an integral around a simple-closed contour by an integral around a circle.

**Theorem 1.4 Shrinking Contour Theorem**

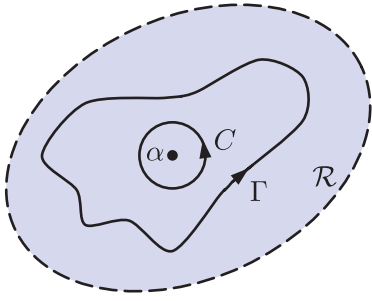
Let  $\mathcal{R}$  be a simply connected region, let  $\Gamma$  be a simple-closed contour in  $\mathcal{R}$ , let  $\alpha$  be a point inside  $\Gamma$ , and let  $f$  be a function that is analytic on  $\mathcal{R} - \{\alpha\}$ . Then

$$\int_{\Gamma} f(z) dz = \int_C f(z) dz,$$

where  $C$  is any circle lying inside  $\Gamma$  with centre  $\alpha$ .



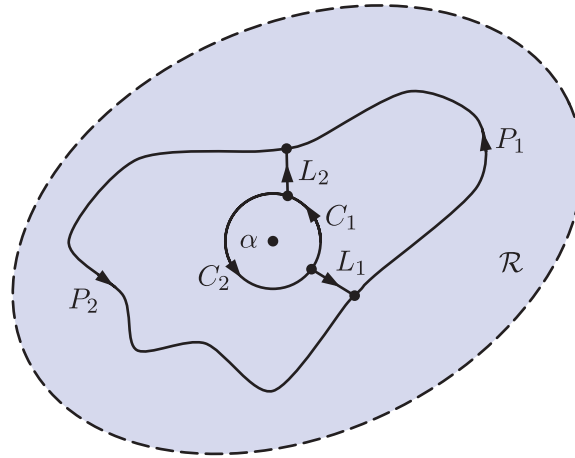
**Figure 1.10** Two contours  $\Gamma_1$  and  $\Gamma_2$  with the same endpoints



**Figure 1.11** A circle  $C$  inside  $\Gamma$

A suitable contour  $\Gamma$  and circle  $C$  are illustrated in Figure 1.11. The purpose of replacing an integral along a simple-closed contour with an integral along a circle will emerge in Section 2.

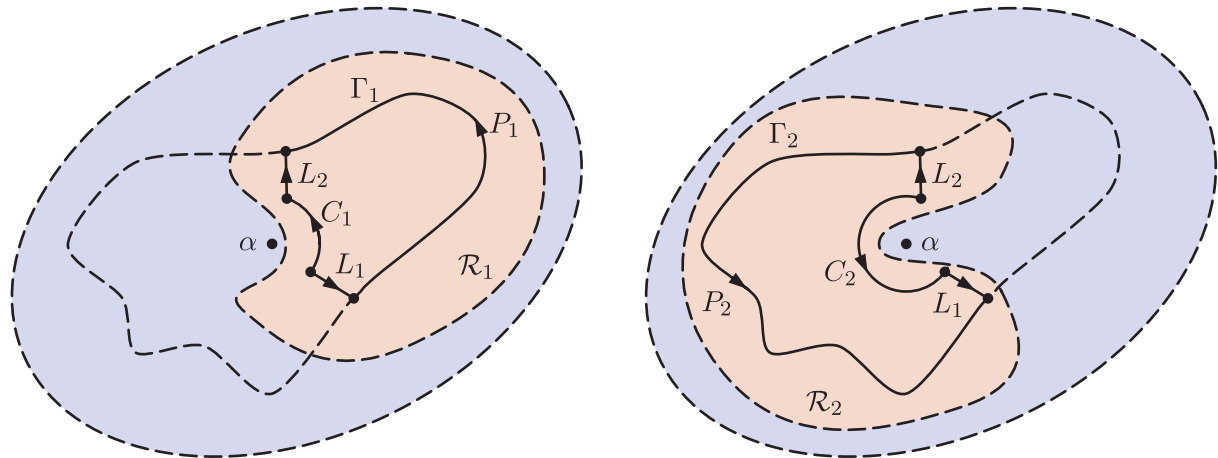
**Proof** Draw two line-segment contours  $L_1$  and  $L_2$ , which do not intersect each other, from the circle  $C$  to the contour  $\Gamma$ , thereby dividing  $\Gamma$  into two parts  $P_1$  and  $P_2$ , and  $C$  into two parts  $C_1$  and  $C_2$ , as shown in Figure 1.12.



**Figure 1.12** Line segments  $L_1$  and  $L_2$  used to split each of  $\Gamma$  and  $C$  into two parts

Let  $\mathcal{R}_1$  be a simply connected region contained in  $\mathcal{R}$  that contains the contour  $\Gamma_1 = L_1 + P_1 + \tilde{L}_2 + \tilde{C}_1$ , but does not contain the point  $\alpha$ ; see Figure 1.13(a). Such a region can be chosen because  $\mathcal{R}$  is simply connected (in contrast, if  $\mathcal{R}$  had a hole in it, inside  $\Gamma_1$ , then no such simply connected region  $\mathcal{R}_1$  could be chosen).

Similarly, let  $\mathcal{R}_2$  be a simply connected region contained in  $\mathcal{R}$  that contains the contour  $\Gamma_2 = L_2 + P_2 + \tilde{L}_1 + \tilde{C}_2$ , but does not contain the point  $\alpha$ ; see Figure 1.13(b).



(a)

(b)

**Figure 1.13** (a) The contour  $\Gamma_1 = L_1 + P_1 + \tilde{L}_2 + \tilde{C}_1$  (b) The contour  $\Gamma_2 = L_2 + P_2 + \tilde{L}_1 + \tilde{C}_2$

Applying Cauchy's Theorem to the function  $f$  around the contour  $\Gamma_1$  in the region  $\mathcal{R}_1$ , we obtain

$$\int_{L_1} f(z) dz + \int_{P_1} f(z) dz + \int_{\tilde{L}_2} f(z) dz + \int_{\tilde{C}_1} f(z) dz = 0.$$

Similarly, applying Cauchy's Theorem to the function  $f$  around the contour  $\Gamma_2$  in the region  $\mathcal{R}_2$ , we obtain

$$\int_{L_2} f(z) dz + \int_{P_2} f(z) dz + \int_{\tilde{L}_1} f(z) dz + \int_{\tilde{C}_2} f(z) dz = 0.$$

If we add these two equations, the integrals along  $L_1$  and  $\tilde{L}_1$  cancel, since

$$\int_{\tilde{L}_1} f(z) dz = - \int_{L_1} f(z) dz,$$

by the Reverse Contour Theorem (Theorem 2.3 of Unit B1), as do the integrals along  $L_2$  and  $\tilde{L}_2$ . We obtain

$$\int_{P_1} f(z) dz + \int_{P_2} f(z) dz + \int_{\tilde{C}_1} f(z) dz + \int_{\tilde{C}_2} f(z) dz = 0;$$

that is,

$$\int_{P_1} f(z) dz + \int_{P_2} f(z) dz = \int_{C_1} f(z) dz + \int_{C_2} f(z) dz.$$

Since  $\Gamma = P_1 + P_2$  and  $C = C_1 + C_2$ , we see that

$$\int_{\Gamma} f(z) dz = \int_C f(z) dz. \quad \blacksquare$$

### Remark

The technique used here of introducing extra contours in order to apply Cauchy's Theorem is a standard technique in complex analysis. It will be used in Unit C1 in the proof of the Residue Theorem.

### Exercise 1.6

Use the Shrinking Contour Theorem to evaluate the integral

$$\int_{\Gamma} \frac{1}{z} dz,$$

where  $\Gamma$  is the ellipse  $\frac{x^2}{9} + \frac{y^2}{4} = 1$ .

## Further exercises

### Exercise 1.7

State whether or not each of the following regions is simply connected. Justify your answers.

- (a)  $\{z : 2 < |z - 3| < 4\}$       (b)  $\{z : |z| > 0\}$       (c)  $\{z : -1 < \operatorname{Re} z < 1\}$   
 (d)  $\{z : -\pi < \operatorname{Arg} z < \pi\}$       (e) The domain of the tangent function

### Exercise 1.8

State which of the following paths  $\Gamma$  are simple-closed. For those that are not, explain why not.

- (a)  $\Gamma : \gamma(t) = e^{it} \ (t \in [0, \pi])$       (b)  $\Gamma : \gamma(t) = e^{it} \ (t \in [0, 2\pi])$   
 (c)  $\Gamma : \gamma(t) = e^{it} \ (t \in [0, 4\pi])$       (d)  $\Gamma : \gamma(t) = te^{it} \ (t \in [0, 2\pi])$

### Exercise 1.9

For each of the following integrals, find a simply connected region containing  $\Gamma = \{z : |z| = 2\}$  on which Cauchy's Theorem can be applied to show that the integral has value zero.

- (a)  $\int_{\Gamma} \sin z \, dz$       (b)  $\int_{\Gamma} \tan(z/2) \, dz$       (c)  $\int_{\Gamma} \frac{1}{z - \pi} \, dz$   
 (d)  $\int_{\Gamma} ((z^3 + 3z - 2)e^z + \operatorname{Log}(z + 3i)) \, dz$

### Exercise 1.10

Explain why Cauchy's Theorem is not appropriate for evaluating each of the following integrals.

- (a)  $\int_{\Gamma} \sec z \, dz$ , where  $\Gamma = \{z : |z| = 2\}$ .  
 (b)  $\int_{\Gamma} \operatorname{Log}(1 + z) \, dz$ , where  $\Gamma = \{z : |z| = 1\}$ .  
 (c)  $\int_{\Gamma} \frac{1}{z - 1} \, dz$ , where  $\Gamma = \{z : |z| = 3\}$ .  
 (d)  $\int_{\Gamma} e^z \, dz$ , where  $\Gamma$  has parametrisation  $\gamma(t) = (1 - t) + it \ (t \in [0, 1])$ .

### Exercise 1.11

For each of the following pairs of integrals, explain why  $I_1$  and  $I_2$  are equal. (Do not evaluate the integrals.)

$$\begin{aligned}
\text{(a)} \quad I_1 &= \int_{\Gamma_1} z e^z dz, \quad \Gamma_1 : \gamma_1(t) = it \quad (t \in [0, 1]) \\
I_2 &= \int_{\Gamma_2} z e^z dz, \quad \Gamma_2 : \gamma_2(t) = \frac{1}{2}i + \frac{1}{2}e^{it} \quad (t \in [-\pi/2, \pi/2]) \\
\text{(b)} \quad I_1 &= \int_{\Gamma_1} \frac{\text{Log } z}{z-3} dz, \quad \Gamma_1 = \{z : |z-4| = 2\} \\
I_2 &= \int_{\Gamma_2} \frac{\text{Log } z}{z-3} dz, \quad \Gamma_2 = \{z : |z-3| = \frac{1}{2}\}
\end{aligned}$$

### Cauchy's legacy

Augustin-Louis Cauchy, whom you encountered in Book A, is rightly regarded as the foremost founder of complex analysis. From 1814 until the end of his life in 1857, he produced a huge array of papers on complex functions, including *Mémoire sur les intégrals définies, prises entre des limites imaginaires* (1825), regarded by many as his masterwork. It was this paper that contained the first version of Cauchy's Theorem and other integral theorems attributed to Cauchy that you will meet in this module.

Cauchy's influence is summed up in an obituary written in 1857 by the French mathematician Charles Auguste Briot (1817–1882):

Young geometers who have the courage to read his works in detail and with care will find them to be a mine of ideas, with rich veins of discoveries and insights to follow through on and to bring up to date.

(Belhoste, 1991, p. 212, cited in Bottazzini and Gray, 2013, p. 212)

From a contemporary perspective, Cauchy's writing can seem at times obscure and misguided, if illuminated by brilliance. This clumsiness in Cauchy's exposition is a reflection of the subtleties of the concepts underpinning complex analysis. However, Briot's recommendation proved to be true, as generations of mathematicians followed Cauchy's lead, refining the subject to its present polished state.

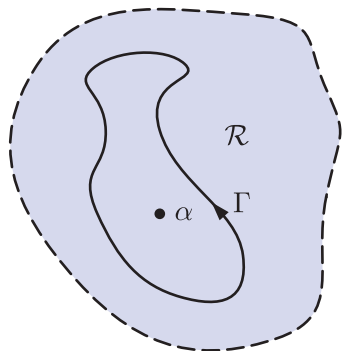
## 2 Cauchy's Integral Formula

After working through this section, you should be able to:

- state and apply Cauchy's Integral Formula
- state and apply Liouville's Theorem
- understand the Fundamental Theorem of Algebra.

## 2.1 Introducing Cauchy's Integral Formula

In this subsection we present a remarkable result which expresses the values of an analytic function  $f$  *inside* a simple-closed contour  $\Gamma$  in terms of the values of  $f$  *on*  $\Gamma$ . The formula involves an integral, and it can also be used in reverse to evaluate integrals, as you will see. The theorem will be proved in Subsection 2.2, and some features of it are illustrated in Figure 2.1.



**Figure 2.1** A point  $\alpha$  inside the simple-closed contour  $\Gamma$

### Theorem 2.1 Cauchy's Integral Formula

Let  $\mathcal{R}$  be a simply connected region, let  $\Gamma$  be a simple-closed contour in  $\mathcal{R}$ , and let  $f$  be a function that is analytic on  $\mathcal{R}$ . Then

$$f(\alpha) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)}{z - \alpha} dz,$$

for any point  $\alpha$  inside  $\Gamma$ .

Cauchy's Integral Formula has great theoretical importance, in that the representation of  $f(\alpha)$  in terms of an integral can be used to find properties of the function  $f$ . For example, we will use it to derive a formula for the derivative  $f'$  in Section 3, and to prove Taylor's Theorem in the next unit.

More important for our present purposes are the practical consequences of Cauchy's Integral Formula. In one direction, it tells us that we can find the value of  $f(\alpha)$  by integrating the function

$$z \mapsto \frac{f(z)}{z - \alpha}$$

around the contour  $\Gamma$  (recall our convention from Subsection 1.3 that  $\Gamma$  is assumed to be traversed once anticlockwise). Often, however, it is more useful to reverse the procedure and write Cauchy's Integral Formula as

$$\int_{\Gamma} \frac{f(z)}{z - \alpha} dz = 2\pi i f(\alpha),$$

and then evaluate the integral using the value of  $f(\alpha)$ . The following examples should make this method for evaluating integrals clear.

### Example 2.1

Evaluate

$$\int_{\Gamma} \frac{e^z}{z - 1} dz,$$

where  $\Gamma$  is the circle  $\{z : |z| = 2\}$ .

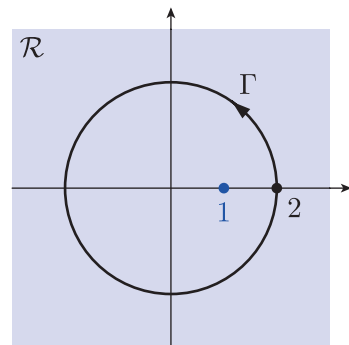
### Solution

We use Cauchy's Integral Formula with  $f(z) = e^z$ ,  $\alpha = 1$  and  $\mathcal{R} = \mathbb{C}$ . We must first check that the hypotheses of the theorem are satisfied.

Certainly  $\mathcal{R}$  is simply connected,  $\Gamma$  is a simple-closed contour in  $\mathcal{R}$ , and  $\alpha$  lies inside  $\Gamma$  (see Figure 2.2). Also,  $f(z) = e^z$  is analytic on  $\mathcal{R}$ .

It follows from Cauchy's Integral Formula that

$$\begin{aligned}\int_{\Gamma} \frac{e^z}{z-1} dz &= 2\pi i f(1) \\ &= 2\pi i e^1 \\ &= 2\pi e i.\end{aligned}$$



**Figure 2.2** The circle  $\Gamma = \{z : |z| = 2\}$

### Exercise 2.1

Use Cauchy's Integral Formula to evaluate the following integrals.

- (a)  $\int_{\Gamma} \frac{\sin z}{z+i} dz$ , where  $\Gamma$  is the circle  $\{z : |z| = 2\}$ .
- (b)  $\int_{\Gamma} \frac{3z}{z+1} dz$ , where  $\Gamma$  is the circle  $\{z : |z-3| = 5\}$ .

When the integrand is not as simple as those considered above, we need to be careful in choosing the function  $f$ , as the following example illustrates.

### Example 2.2

Evaluate

$$\int_{\Gamma} \frac{z^2 + 3}{z(z-2)} dz,$$

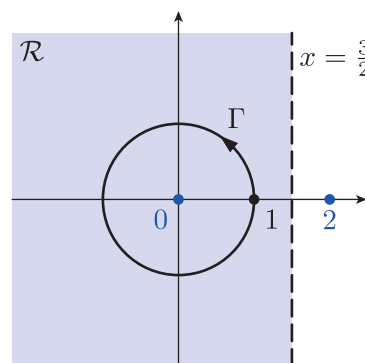
where  $\Gamma$  is the circle  $\{z : |z| = 1\}$ .

### Solution

Looking at the denominator, we see that 0 and 2 are the only points where the integrand is not defined. Of these two points, only 0 lies inside  $\Gamma$ . We therefore take  $f(z) = (z^2 + 3)/(z-2)$  and  $\alpha = 0$ , and let  $\mathcal{R}$  be any simply connected region that contains  $\Gamma$  but not the point 2; for example,  $\mathcal{R} = \{z : \operatorname{Re} z < \frac{3}{2}\}$ . Then  $\Gamma$  is a simple-closed contour in  $\mathcal{R}$ , and  $\alpha$  lies inside  $\Gamma$  (see Figure 2.3). Also,  $f$  is analytic on  $\mathcal{R}$ .

It follows from Cauchy's Integral Formula that

$$\begin{aligned}\int_{\Gamma} \frac{z^2 + 3}{z(z-2)} dz &= \int_{\Gamma} \frac{(z^2 + 3)/(z-2)}{z} dz \\ &= \int_{\Gamma} \frac{f(z)}{z-0} dz \\ &= 2\pi i f(0) \\ &= 2\pi i \times \left(-\frac{3}{2}\right) = -3\pi i.\end{aligned}$$



**Figure 2.3** The circle  $\Gamma = \{z : |z| = 1\}$  in the region  $\mathcal{R}$

## Exercise 2.2

Use Cauchy's Integral Formula to evaluate the following integrals.

- (a)  $\int_{\Gamma} \frac{e^{3z}}{z^2 - 4} dz$ , where  $\Gamma = \{z : |z - 1| = 2\}$ .
- (b)  $\int_{\Gamma} \frac{\cos 2z}{z(z^2 + 4)} dz$ , where  $\Gamma$  is the square contour with vertices  $1 + i$ ,  $-1 + i$ ,  $-1 - i$ ,  $1 - i$ .

As we have seen, Cauchy's Integral Formula can be used to evaluate integrals of the form

$$\int_{\Gamma} \frac{f(z)}{z - \alpha} dz,$$

where  $f$  is a function that is analytic on a simply connected region containing  $\Gamma$ . We now look at how to extend the range of applicability of Cauchy's Integral Formula by using *partial fractions*.

Consider the integral

$$\int_{\Gamma} \frac{e^{2z}}{z^2 + 1} dz, \quad \text{where } \Gamma = \{z : |z - 1| = 2\}. \quad (2.1)$$

This is not in a suitable form to apply Cauchy's Integral Formula because the denominator  $z^2 + 1 = (z + i)(z - i)$  has two zeros,  $-i$  and  $i$ , both of which lie inside  $\Gamma$ . However, by splitting the rational expression  $1/(z^2 + 1)$  into a sum of two simpler rational expressions, we can write the integral as a sum of two integrals that *are* in a form suitable for Cauchy's Integral Formula.

To do this, we write

$$\frac{1}{z^2 + 1} = \frac{1}{(z + i)(z - i)} = \frac{A}{z + i} + \frac{B}{z - i},$$

where  $A$  and  $B$  are complex numbers to be determined. One way to find  $A$  and  $B$  is by the method of *equating coefficients*, as follows.

First we multiply both sides by  $z^2 + 1$ , to give

$$1 = A(z - i) + B(z + i). \quad (2.2)$$

By equating the coefficients of  $z$  and constants, we obtain

$$z : 0 = A + B,$$

$$1 : 1 = -Ai + Bi.$$

Solving these simultaneous equations gives  $A = i/2$  and  $B = -i/2$ . Thus

$$\frac{1}{z^2 + 1} = \frac{i/2}{z + i} - \frac{i/2}{z - i}. \quad (2.3)$$

Alternatively, we can find  $A$  and  $B$  by substituting suitable values of  $z$  into equation (2.2) rather than by equating coefficients. For example, substituting  $z = i$  into equation (2.2) gives  $1 = 2Bi$ , so  $B = -i/2$ , and substituting  $z = -i$  gives  $A = i/2$ .

The expression on the right of equation (2.3) is the *partial fraction expansion* of  $1/(z^2 + 1)$ , and the terms

$$\frac{i/2}{z+i} \quad \text{and} \quad -\frac{i/2}{z-i}$$

are the *partial fractions* of  $1/(z^2 + 1)$ . By similar methods, we can find the partial fraction expansion of any rational expression  $1/p(z)$ , where  $p(z)$  is a polynomial expression that can be written as a product of distinct linear factors.

We can now use the partial fraction expansion of  $1/(z^2 + 1)$  together with Cauchy's Integral Formula to evaluate the integral (2.1). Before we do this, however, you should try the following exercise, the results of which will be used in Exercise 2.4 and Example 2.4.

### Exercise 2.3

Find the partial fraction expansions of the following expressions.

(a)  $\frac{1}{z^2 - z}$       (b)  $\frac{1}{z(z - 2)}$

We now show how partial fractions can be used in conjunction with Cauchy's Integral Formula.

### Example 2.3

Evaluate

$$\int_{\Gamma} \frac{e^{2z}}{z^2 + 1} dz,$$

where  $\Gamma$  is the circle  $\{z : |z - 1| = 2\}$ .

### Solution

Using the partial fractions obtained in equation (2.3), we have

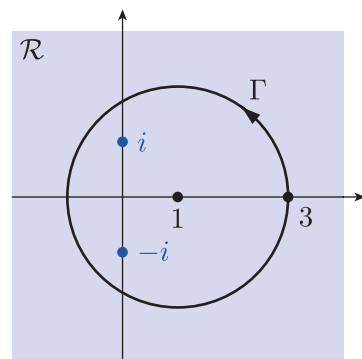
$$\int_{\Gamma} \frac{e^{2z}}{z^2 + 1} dz = \frac{i}{2} \int_{\Gamma} \frac{e^{2z}}{z+i} dz - \frac{i}{2} \int_{\Gamma} \frac{e^{2z}}{z-i} dz.$$

We now use Cauchy's Integral Formula with  $f(z) = e^{2z}$ ,  $\mathcal{R} = \mathbb{C}$  and  $\alpha = -i$  and  $i$  (in turn). Then  $\mathcal{R}$  is simply connected,  $\Gamma$  is a simple-closed contour in  $\mathcal{R}$ ,  $-i$  and  $i$  lie inside  $\Gamma$ , and  $f$  is analytic on  $\mathcal{R}$  (see Figure 2.4). Thus

$$\int_{\Gamma} \frac{e^{2z}}{z+i} dz = 2\pi i f(-i) = 2\pi i e^{-2i}$$

and

$$\int_{\Gamma} \frac{e^{2z}}{z-i} dz = 2\pi i f(i) = 2\pi i e^{2i}.$$



**Figure 2.4** The points  $-i$  and  $i$  inside the circle  $\Gamma = \{z : |z - 1| = 2\}$

Putting all this together, we obtain

$$\begin{aligned}\int_{\Gamma} \frac{e^{2z}}{z^2 + 1} dz &= \frac{i}{2} \times 2\pi i e^{-2i} - \frac{i}{2} \times 2\pi i e^{2i} \\ &= \pi(e^{2i} - e^{-2i}) \\ &= 2\pi i \sin 2, \\ \text{since } \sin z &= (e^{iz} - e^{-iz})/(2i).\end{aligned}$$

### Exercise 2.4

Use Cauchy's Integral Formula to evaluate

$$\int_{\Gamma} \frac{\cos 3z}{z^2 - z} dz,$$

where  $\Gamma$  is the circle  $\{z : |z - 1| = 2\}$ .

The following example illustrates how we can sometimes save effort when partial fractions are involved.

### Example 2.4

Evaluate

$$\int_{\Gamma} \frac{\cos z}{z(z^2 - 4)} dz,$$

where  $\Gamma$  is the circle  $\{z : |z - 2| = 3\}$ .

### Solution

First note that the denominator can be factorised as a product of distinct linear factors:

$$z(z^2 - 4) = z(z - 2)(z + 2).$$

It is tempting to expand  $1/(z(z^2 - 4))$  in partial fractions as  $A/z + B/(z - 2) + C/(z + 2)$  and proceed as in Example 2.3. However, an alternative approach involving a simpler partial fraction expansion is as follows.

The three zeros of the expression  $z(z^2 - 4)$  are 0, 2 and  $-2$ . Of these, 0 and 2 lie inside  $\Gamma$  but  $-2$  lies outside  $\Gamma$ . This suggests combining  $(z + 2)$  with  $\cos z$  to form  $f(z) = (\cos z)/(z + 2)$ . Using partial fractions (see Exercise 2.3(b)) we have

$$\frac{1}{z(z - 2)} = -\frac{1}{2z} + \frac{1}{2(z - 2)},$$

so we can write

$$\int_{\Gamma} \frac{\cos z}{z(z^2 - 4)} dz = -\frac{1}{2} \int_{\Gamma} \frac{f(z)}{z} dz + \frac{1}{2} \int_{\Gamma} \frac{f(z)}{z - 2} dz.$$

We now let  $\mathcal{R}$  be any simply connected region containing  $\Gamma$  but not the point  $-2$ ; for example,  $\mathcal{R} = \{z : \operatorname{Re} z > -\frac{3}{2}\}$  (see Figure 2.5). Applying Cauchy's Integral Formula with  $f(z) = (\cos z)/(z+2)$ , which is analytic on  $\mathcal{R}$ , and  $\alpha = 0$  and  $2$  (in turn), gives

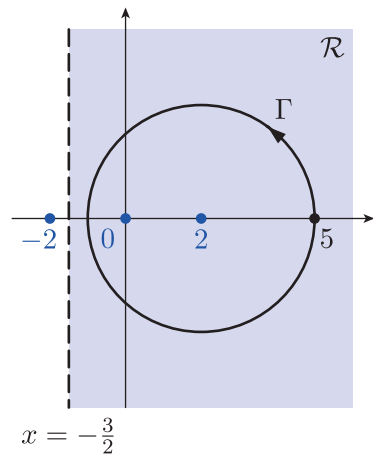
$$\begin{aligned}\int_{\Gamma} \frac{f(z)}{z} dz &= 2\pi i f(0) \\ &= 2\pi i \times \frac{1}{2} = \pi i\end{aligned}$$

and

$$\begin{aligned}\int_{\Gamma} \frac{f(z)}{z-2} dz &= 2\pi i f(2) \\ &= 2\pi i \times \frac{\cos 2}{4} = \frac{\pi}{2} i \cos 2.\end{aligned}$$

Putting all this together, we obtain

$$\begin{aligned}\int_{\Gamma} \frac{\cos z}{z(z^2-4)} dz &= -\frac{1}{2} \times \pi i + \frac{1}{2} \times \frac{\pi}{2} i \cos 2 \\ &= \frac{\pi}{4} (-2 + \cos 2) i.\end{aligned}$$



**Figure 2.5** The points 0 and 2 inside the circle  $\Gamma = \{z : |z-2| = 3\}$

### Exercise 2.5

- (a) Use the partial fraction expansion

$$\frac{1}{z^4-1} = \frac{1}{4} \left( \frac{1}{z-1} - \frac{1}{z+1} + \frac{i}{z-i} - \frac{i}{z+i} \right)$$

to evaluate

$$\int_{\Gamma} \frac{e^z}{z^4-1} dz,$$

where  $\Gamma$  is

- (i) the ellipse  $4x^2 + 9y^2 = 36$
- (ii) the rectangular contour with vertices  $\frac{1}{2} + 2i, -\frac{1}{2} + 2i, -\frac{1}{2} - 2i, \frac{1}{2} - 2i$ .

- (b) Confirm your answer to part (a)(ii) by using the approach of Example 2.4.

Using the approaches just described, we can evaluate *any* integral of the form

$$\int_{\Gamma} \frac{g(z)}{p(z)} dz,$$

where  $\Gamma$  is a simple-closed contour,  $g$  is a function that is analytic on some simply connected region  $\mathcal{R}$  containing  $\Gamma$ , and  $p$  is a polynomial function with *distinct* roots that do not lie on  $\Gamma$ . This is because  $p(z)$  can be

expressed as a product of  $n$  distinct linear factors by the Fundamental Theorem of Algebra, discussed at the end of this section. But we cannot yet evaluate integrals involving polynomial functions with repeated roots in the denominator, such as

$$\int_{\Gamma} \frac{e^z}{(z-1)^2} dz.$$

Methods for dealing with such integrals will be discussed in Section 3, and a general strategy is given in Section 4.

## 2.2 Proof of Cauchy's Integral Formula

In this subsection we prove Cauchy's Integral Formula.

### Theorem 2.1 Cauchy's Integral Formula

Let  $\mathcal{R}$  be a simply connected region, let  $\Gamma$  be a simple-closed contour in  $\mathcal{R}$ , and let  $f$  be a function that is analytic on  $\mathcal{R}$ . Then

$$f(\alpha) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)}{z - \alpha} dz,$$

for any point  $\alpha$  inside  $\Gamma$ .

Before we give a proof, we make two introductory remarks.

#### Remarks

1. A standard method to prove that a complex number  $I$  is zero is to show that, for some positive constant  $K$ ,

$$|I| \leq K\varepsilon, \tag{2.4}$$

for each positive number  $\varepsilon$ . To see why this proves that  $I = 0$ , suppose that  $I \neq 0$ , and take  $\varepsilon = \frac{1}{2}|I|/K$  in inequality (2.4). This gives

$$|I| \leq \frac{1}{2}|I|,$$

which is false. Thus  $I = 0$  after all.

2. During the proof, we will need the fact that

$$\int_C \frac{1}{z - \alpha} dz = 2\pi i,$$

where  $C$  is any circle with centre  $\alpha$  (and radius  $r$ , say). We proved a special case of this result for  $\alpha = 0$ ,  $r = 1$  in Example 2.3 of Unit B1. To prove the general case, we use the standard parametrisation

$$\gamma(t) = \alpha + re^{it} \quad (t \in [0, 2\pi]).$$

Then  $\gamma'(t) = rie^{it}$ , and we have

$$\begin{aligned} \int_C \frac{1}{z - \alpha} dz &= \int_0^{2\pi} \frac{rie^{it}}{re^{it}} dt \\ &= \int_0^{2\pi} i dt = 2\pi i. \end{aligned}$$

We now prove Cauchy's Integral Formula. The basic tools are the Shrinking Contour Theorem (Theorem 1.4) and the Estimation Theorem (Theorem 4.1 of Unit B1).

**Proof of Cauchy's Integral Formula** There are four steps in the proof.

1. Consider the integral

$$\int_{\Gamma} \frac{f(z)}{z - \alpha} dz.$$

By the Shrinking Contour Theorem applied to the function with rule  $f(z)/(z - \alpha)$ , we can replace  $\Gamma$  by any circle  $C$  with centre  $\alpha$ , lying inside  $\Gamma$  (such a circle  $C$  exists, since the inside of  $\Gamma$  is an open set), to obtain

$$\int_{\Gamma} \frac{f(z)}{z - \alpha} dz = \int_C \frac{f(z)}{z - \alpha} dz.$$

The radius  $r$  of  $C$  will be chosen in step 3.

2. Let

$$I = \int_C \frac{f(z)}{z - \alpha} dz - 2\pi i f(\alpha).$$

To prove the theorem, we need to show that the complex number  $I$  is equal to zero.

By Remark 2, we can replace the  $2\pi i$  by

$$\int_C \frac{1}{z - \alpha} dz$$

to give

$$\begin{aligned} I &= \int_C \frac{f(z)}{z - \alpha} dz - f(\alpha) \int_C \frac{1}{z - \alpha} dz \\ &= \int_C \left( \frac{f(z)}{z - \alpha} - \frac{f(\alpha)}{z - \alpha} \right) dz, \\ &= \int_C \frac{f(z) - f(\alpha)}{z - \alpha} dz. \end{aligned}$$

In the second-to-last line we have taken  $f(\alpha)$  inside the integral, which is valid because it is a constant.

3. We now use the Estimation Theorem to give an upper estimate for  $|I|$ . The length of  $C$  is easy to find: it is just  $2\pi r$ , the circumference of the circle. To find an upper estimate for

$$\left| \frac{f(z) - f(\alpha)}{z - \alpha} \right|$$

on  $C$ , we use the fact that  $f$  is continuous at  $\alpha$  (since it is differentiable at  $\alpha$ ). Thus for each  $\varepsilon > 0$ , there is  $\delta > 0$  such that

$$|z - \alpha| < \delta \implies |f(z) - f(\alpha)| < \varepsilon.$$

If we now choose the radius  $r$  to be any positive number less than  $\delta$ , then we can write

$$\left| \frac{f(z) - f(\alpha)}{z - \alpha} \right| = \frac{|f(z) - f(\alpha)|}{|z - \alpha|} < \frac{\varepsilon}{r},$$

for  $z \in C$ . It follows from the Estimation Theorem, with  $M = \varepsilon/r$  and  $L = 2\pi r$ , that

$$|I| \leq \frac{\varepsilon}{r} \times 2\pi r = 2\pi\varepsilon.$$

4. Finally, we use the result of Remark 1. Since  $|I| \leq 2\pi\varepsilon$ , for each positive number  $\varepsilon$ , we obtain  $I = 0$ . This concludes the proof. ■

### Exercise 2.6

Let  $\mathcal{R}$  be a simply connected region, let  $C = \{z : |z - \alpha| = r\}$  be a circle contained in  $\mathcal{R}$ , and let  $f$  be a function that is analytic on  $\mathcal{R}$ . Use Cauchy's Integral Formula and the standard parametrisation of  $C$  to prove that

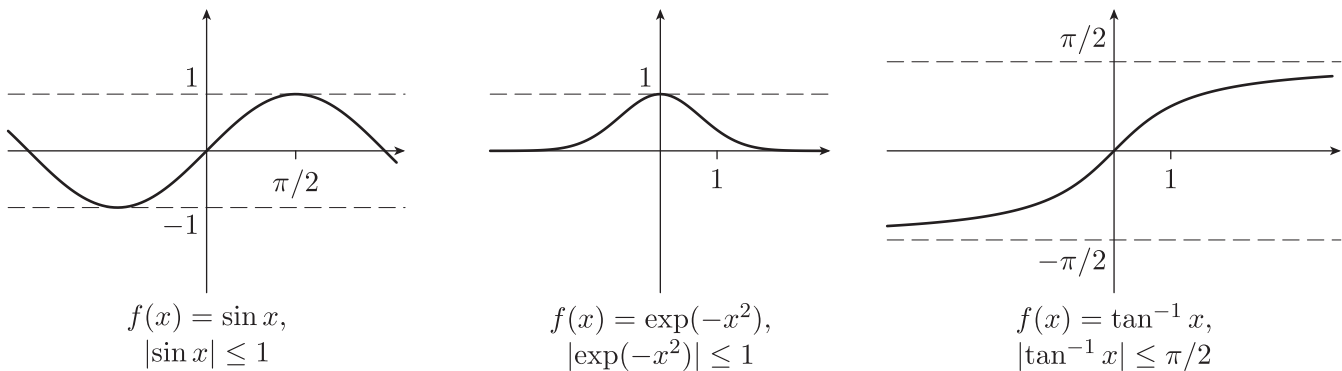
$$f(\alpha) = \frac{1}{2\pi} \int_0^{2\pi} f(\alpha + re^{it}) dt.$$

The result of this exercise is called **Gauss's Mean Value Theorem**. It tells us that the value of  $f$  at the centre of the circle  $C$  is the 'average' of the values of  $f$  on  $C$ .

## 2.3 Liouville's Theorem

We come now to one of the most surprising results in complex analysis, namely Liouville's Theorem. (Liouville is pronounced 'lee-oo-vill', or similar.)

In your study of real functions, you will have met many functions that are differentiable at all values of  $x$ , and which are bounded. Examples of such functions are  $f(x) = \sin x$ ,  $f(x) = \exp(-x^2)$  and  $f(x) = \tan^{-1} x$ , the graphs of which are illustrated in Figure 2.6.



**Figure 2.6** Real functions that are differentiable on the entire real line and are bounded

In complex analysis, however, the class of all bounded functions that are entire (that is, differentiable at all values of  $z$ ) is highly restricted: it consists of only the constant functions!

### Theorem 2.2 Liouville's Theorem

Every bounded entire function is a constant function.

Another way of expressing this result is to say that

if  $f$  is a non-constant entire function, then  $f$  is unbounded.

Before proving Liouville's Theorem, we give two exercises to test your understanding of its statement.

#### Exercise 2.7

Determine what is wrong with the following *false* statement.

'Since  $f(z) = \sin z$  is entire and  $|\sin z| \leq 1$ , for all  $z \in \mathbb{C}$ , the sine function is constant, by Liouville's Theorem.'

#### Exercise 2.8

Use Liouville's Theorem to prove that the function  $f(z) = \exp(i|z|)$  is not an entire function.

Let us now prove Liouville's Theorem. In order to prove that a bounded entire function  $f$  is a constant function, we choose an arbitrary point  $\alpha$  in the complex plane and show that  $f(\alpha) = f(0)$ . To do this, we use Cauchy's Integral Formula to write

$$f(\alpha) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - \alpha} dz,$$

where  $C$  is a circle with centre 0 that contains  $\alpha$  inside.

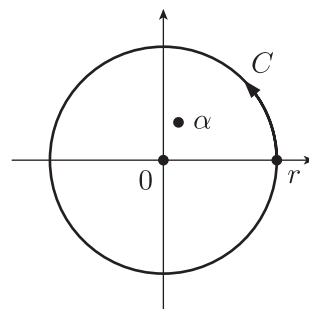
If the radius of  $C$  is large, then the points  $\alpha$  and 0 (as seen from  $C$ ) will look close. We use this fact to show that  $f(\alpha) - f(0) = 0$ .

**Proof of Liouville's Theorem** The proof, which may be omitted on a first reading, is in four steps.

Suppose that  $f$  is a bounded entire function.

1. Let  $\alpha$  be any point of  $\mathbb{C}$ , and let  $C$  be a circle with centre 0 and radius  $r$ , where  $r > |\alpha|$  (see Figure 2.7). Since  $f$  is analytic on  $\mathbb{C}$ , and  $\mathbb{C}$  is simply connected, we can apply Cauchy's Integral Formula to give

$$f(\alpha) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - \alpha} dz \quad \text{and} \quad f(0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z} dz.$$



**Figure 2.7** Point  $\alpha$  inside the circle  $C$  with centre 0 and radius  $r$

Thus

$$\begin{aligned} f(\alpha) - f(0) &= \frac{1}{2\pi i} \int_C \left( \frac{1}{z - \alpha} - \frac{1}{z} \right) f(z) dz \\ &= \frac{\alpha}{2\pi i} \int_C \frac{f(z)}{z(z - \alpha)} dz. \end{aligned}$$

2. We now estimate this integral.

Since, by assumption,  $f$  is bounded, there exists a number  $K$  such that

$$|f(z)| \leq K, \quad \text{for all } z \in \mathbb{C}.$$

Also, for each  $z$  on  $C$ , we have  $|z| = r$ , so, by the backwards form of the Triangle Inequality,

$$|z - \alpha| \geq |z| - |\alpha| = r - |\alpha|, \quad \text{for } z \in C.$$

It follows that

$$\left| \frac{f(z)}{z(z - \alpha)} \right| \leq \frac{K}{r(r - |\alpha|)}, \quad \text{for } z \in C.$$

Then, using the Estimation Theorem, with  $M = K/(r(r - |\alpha|))$  and  $L = 2\pi r$ , we obtain

$$\begin{aligned} |f(\alpha) - f(0)| &= \left| \frac{\alpha}{2\pi i} \int_C \frac{f(z)}{z(z - \alpha)} dz \right| \\ &\leq \frac{|\alpha|}{2\pi} \times \frac{K}{r(r - |\alpha|)} \times 2\pi r \\ &= \frac{K|\alpha|}{r - |\alpha|}. \end{aligned}$$

3. The inequality just obtained is valid for any value of  $r > |\alpha|$ . In particular, given any positive number  $\varepsilon$ , we can choose  $r$  to be sufficiently large so that

$$|f(\alpha) - f(0)| \leq \frac{K|\alpha|}{r - |\alpha|} < \varepsilon.$$

To do this, we observe that

$$\begin{aligned} \frac{K|\alpha|}{r - |\alpha|} < \varepsilon &\iff r - |\alpha| > |\alpha|K/\varepsilon \\ &\iff r > |\alpha|(1 + K/\varepsilon). \end{aligned}$$

It follows that if  $r > |\alpha|(1 + K/\varepsilon)$ , then  $|f(\alpha) - f(0)| < \varepsilon$ .

4. We now see (using Remark 1 after Theorem 2.1 of Subsection 2.2) that

$$f(\alpha) - f(0) = 0,$$

so  $f(\alpha) = f(0)$ . Since this is true for any point  $\alpha \in \mathbb{C}$ , we deduce that  $f$  is a constant function. ■

### Origin of Liouville's Theorem

In 1844 the French mathematician Joseph Liouville (1809–1882) announced the theorem that is now known as Liouville's Theorem. However, he stated the theorem for only a certain class of analytic functions known as *doubly periodic functions*. He wrote:

If a (single-valued) function is doubly periodic, and if one recognises that it never becomes infinite, one can, from this alone, affirm that it reduces to a constant.

(Bottazzini and Gray, 2013, p. 179)

It seems that Liouville did not appreciate the full significance of his result, because although he lectured on the topic, he never published a proof of the theorem. On learning of Liouville's idea, however, Cauchy immediately recognised its importance, and in 1844 produced several proofs of the general version of the theorem that we use today. Cauchy wrote:

If a single-valued function reduces to a determinate constant  $F$  for every infinite value of  $z$ , then it will reduce to the same constant value when the variable  $z$  has any finite value.

(Bottazzini and Gray, 2013, p. 180)

This somewhat cryptic statement is essentially the same as the result that we call Liouville's Theorem, but uses language typical of Cauchy from that period.

There is a remarkable generalisation of Liouville's Theorem due to the French mathematician Émile Picard (1856–1941). This result, which is called *Picard's Little Theorem*, published in 1879, says that any entire function whose image set omits two points in the complex plane is constant.

To help appreciate this theorem, consider the function  $\exp$ . The image set of  $\exp$  omits the point 0, because  $\exp z$  is never 0. Since  $\exp$  is not a constant function, we see from the theorem that the image set of  $\exp$  must be  $\mathbb{C} - \{0\}$  (a fact we knew already).

Picard's Little Theorem is one of the jewels of complex analysis, but it is not an assessed part of this module, so you cannot quote it in your work – unless you prove it first!



Joseph Liouville

We finish this subsection with an important application of Liouville's Theorem. Recall that a *zero* of a function  $f$  is a value  $z$  for which  $f(z) = 0$ .

### Theorem 2.3 Fundamental Theorem of Algebra

Every non-constant polynomial function has at least one zero.

**Proof** Suppose, in order to reach a contradiction, that  $p$  is a polynomial function of degree  $n$  (where  $n > 0$ ) without any zeros. Let  $a_n$  be the coefficient of  $z^n$  in  $p(z)$ . By dividing  $p(z)$  through by  $a_n$  (which does not introduce any zeros), we can assume that  $a_n = 1$ , in which case

$$p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0, \quad (2.5)$$

where  $a_0, a_1, \dots, a_{n-1} \in \mathbb{C}$ .

Let  $M = \max\{|a_0|, |a_1|, \dots, |a_{n-1}|\}$ . If  $M = 0$ , then  $p(z) = z^n$ , which has a zero at  $z = 0$ , so we can assume that  $M > 0$ . We will show that

$$|p(z)| \geq 1, \quad \text{for } |z| \geq 1 + M. \quad (2.6)$$

To do this, we first apply the Triangle Inequality to obtain

$$\begin{aligned} |a_{n-1}z^{n-1} + \cdots + a_1z + a_0| &\leq |a_{n-1}z^{n-1}| + \cdots + |a_1z| + |a_0| \\ &\leq M(|z|^{n-1} + \cdots + |z| + 1). \\ &= M\left(\frac{|z|^n - 1}{|z| - 1}\right), \quad \text{for } |z| \neq 1. \end{aligned}$$

Now suppose that  $|z| \geq 1 + M$ . Then  $|z| - 1 \geq M$ , so

$$M\left(\frac{|z|^n - 1}{|z| - 1}\right) \leq M\left(\frac{|z|^n - 1}{M}\right) = |z|^n - 1.$$

Hence

$$|a_{n-1}z^{n-1} + \cdots + a_1z + a_0| \leq |z|^n - 1. \quad (2.7)$$

Next we apply the backwards form of the Triangle Inequality to give

$$\begin{aligned} |p(z)| &= |z^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0| \\ &\geq |z|^n - |a_{n-1}z^{n-1} + \cdots + a_1z + a_0|. \end{aligned} \quad (2.8)$$

This last expression is greater than or equal to 1, by inequality (2.7), so we have shown that  $|p(z)| \geq 1$ , for  $|z| \geq 1 + M$ , as required.

Let us now define  $f(z) = 1/p(z)$ , which is an entire function because  $p$  has no zeros. By inequality (2.6), we have

$$|f(z)| = \frac{1}{|p(z)|} \leq 1, \quad \text{for } |z| \geq 1 + M.$$

Furthermore, by the Boundedness Theorem (Theorem 5.3 of Unit A3),  $f$  is also bounded on the compact set  $\{z : |z| \leq 1 + M\}$  because it is analytic and hence continuous on  $\mathbb{C}$ . Therefore  $f$  is a bounded function (on the whole of  $\mathbb{C}$ ).

We have established that  $f$  is bounded and entire, so it is constant, by Liouville's Theorem. Therefore  $p$  is constant too, which is a contradiction, as we assumed otherwise. Hence, contrary to our assumption,  $p$  has at least one zero. ■

It is remarkable that techniques from complex analysis can be used to prove this central theorem in algebra. In fact, the proof shows that not only does the polynomial function  $p$  (given by equation (2.5)) have at least one zero, but any zero of  $p$  has modulus less than

$$1 + M = 1 + \max\{|a_0|, |a_1|, \dots, |a_{n-1}|\}.$$

The following corollary to the Fundamental Theorem of Algebra demonstrates that a polynomial function of degree  $n \geq 1$  has not one but  $n$  zeros, possibly with some 'repeated'. To prove the corollary, we use the Geometric Series Identity (Theorem 1.3 of Unit A1) in the form

$$z^k - \alpha^k = (z - \alpha)(z^{k-1} + z^{k-2}\alpha + z^{k-3}\alpha^2 + \cdots + \alpha^{k-1})$$

to observe that  $z^k - \alpha^k$  can be written as the product of  $(z - \alpha)$  with a polynomial expression of degree  $k - 1$ .

### Corollary

Any polynomial function  $p$  of degree  $n \geq 1$  can be expressed as

$$p(z) = a(z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_n),$$

where  $a$  is a non-zero complex number and  $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{C}$  are all zeros of  $p$ , some of which may be repeated.

**Proof** Let us write  $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ , where  $a_0, a_1, \dots, a_{n-1}, a_n \in \mathbb{C}$  and  $a_n \neq 0$ . By the Fundamental Theorem of Algebra,  $p$  has a zero  $\alpha_1$ , so  $p(\alpha_1) = 0$ . Hence

$$\begin{aligned} p(z) &= p(z) - p(\alpha_1) \\ &= a_n(z^n - \alpha_1^n) + a_{n-1}(z^{n-1} - \alpha_1^{n-1}) + \cdots + a_1(z - \alpha_1). \end{aligned}$$

Using the observation made just before the corollary, applied to each of the terms  $z^k - \alpha_1^k$ ,  $k = 1, 2, \dots, n$ , we can write  $p(z) = (z - \alpha_1)q(z)$ , where the 'quotient' function  $q$  is a polynomial function of degree  $n - 1$ . We can then repeat this argument with  $q$  in place of  $p$ , and carry on in this way obtaining successive factors  $(z - \alpha_2), (z - \alpha_3), \dots$ , until we reach a quotient function of degree 0, a constant function, with value  $a$  say. As a result, we obtain the equation

$$p(z) = a(z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_n).$$

Clearly each value  $\alpha_k$  is a zero of  $p$ , and  $a$  is non-zero (in fact  $a = a_n$ , the coefficient of  $z^n$  in  $p(z)$ ). ■

The next exercise is about another interesting corollary to the Fundamental Theorem of Algebra.

### Exercise 2.9

Prove that the image set of any non-constant polynomial function is the entire complex plane.

(Hint: For a non-constant polynomial function  $p$  and a point  $w$  in  $\mathbb{C}$ , consider the function  $f(z) = p(z) - w$ .)

## Further exercises

### Exercise 2.10

Evaluate each of the following integrals by using either Cauchy's Integral Formula or Cauchy's Theorem, as appropriate. In each case,  $\Gamma = \{z : |z - 1| = 2\}$ .

- (a)  $\int_{\Gamma} \frac{e^{i\pi z/2}}{z-1} dz$     (b)  $\int_{\Gamma} \frac{z^3}{z-2} dz$     (c)  $\int_{\Gamma} \frac{z+4}{z-4} dz$   
 (d)  $\int_{\Gamma} \frac{\sin z}{z-i} dz$     (e)  $\int_{\Gamma} \frac{z^2}{z^2-4} dz$

### Exercise 2.11

Use partial fractions and Cauchy's Integral Formula to evaluate the following integrals. In each case,  $\Gamma = \{z : |z - 1| = 3\}$ .

- (a)  $\int_{\Gamma} \frac{2z}{z^2-1} dz$     (b)  $\int_{\Gamma} \frac{\sin 2z}{z^2+1} dz$     (c)  $\int_{\Gamma} \frac{6 \cosh z}{z(z^2-9)} dz$

### Exercise 2.12

- (a) Use Liouville's Theorem to show that if  $f$  is an entire function and  $K$  is a positive constant such that

$$|f(z)| \geq K, \quad \text{for all } z \in \mathbb{C},$$

then  $f$  is a constant function.

- (b) By finding a counterexample, show that the result in part (a) is not valid if the condition on  $|f(z)|$  is replaced by

$$|f(z)| > 0, \quad \text{for all } z \in \mathbb{C}.$$

### History of the Fundamental Theorem of Algebra

The Fundamental Theorem of Algebra has a long and complicated history. One of the earliest statements of the theorem appeared without proof in 1629 in a book by the French-born mathematician Albert Girard (1595–1632). Attempts were made to prove the theorem in the following two centuries by numerous mathematicians, including d'Alembert, Euler, Laplace and Gauss. There was much dispute about the validity of these proofs, and none of them were rigorous by today's standards. It is a testament to the effectiveness of complex analysis that through the development of that subject, the first rigorous proofs of this key theorem began to emerge in the nineteenth century.

### 3 Cauchy's Derivative Formulas

After working through this section, you should be able to:

- state and apply Cauchy's  $n$ th Derivative Formula
- state and apply the Analyticity of Derivatives Theorem.

#### 3.1 Cauchy's First Derivative Formula

In the previous section you saw that if  $f$  is an analytic function on a simply connected region  $\mathcal{R}$ , and if  $\Gamma$  is a simple-closed contour in  $\mathcal{R}$ , then Cauchy's Integral Formula holds:

$$f(\alpha) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)}{z - \alpha} dz, \quad (3.1)$$

for any point  $\alpha$  inside  $\Gamma$ . We now show that there is a similar formula for the first derivative  $f'$ , namely

$$f'(\alpha) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)}{(z - \alpha)^2} dz.$$

Observe that this is the result you would get if you argued as follows. In equation (3.1),  $\alpha$  is a fixed point inside  $\Gamma$ , but suppose that we now think of  $\alpha$  as a (complex) variable, and differentiate each side of equation (3.1) with respect to  $\alpha$ . We obtain

$$\begin{aligned} f'(\alpha) &= \frac{d}{d\alpha} \left( \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)}{z - \alpha} dz \right) \\ &= \frac{1}{2\pi i} \int_{\Gamma} \frac{d}{d\alpha} \left( \frac{1}{z - \alpha} \right) f(z) dz \\ &= \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)}{(z - \alpha)^2} dz. \end{aligned}$$

This is the right answer, but the reasoning is faulty since we cannot just 'differentiate under the integral sign' without justification. We give a full proof of the formula for  $f'(\alpha)$  at the end of this subsection.

The formal statement of the formula is as follows.

#### Theorem 3.1 Cauchy's First Derivative Formula

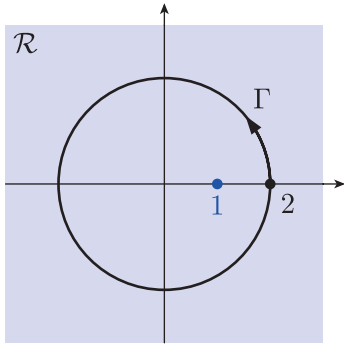
Let  $\mathcal{R}$  be a simply connected region, let  $\Gamma$  be a simple-closed contour in  $\mathcal{R}$ , and let  $f$  be a function that is analytic on  $\mathcal{R}$ . Then

$$f'(\alpha) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)}{(z - \alpha)^2} dz,$$

for any point  $\alpha$  inside  $\Gamma$ .

We can use Cauchy's First Derivative Formula to evaluate certain integrals with square factors in their denominators, by writing

$$\int_{\Gamma} \frac{f(z)}{(z - \alpha)^2} dz = 2\pi i f'(\alpha).$$



**Figure 3.1** The point 1 inside the circle  $\Gamma = \{z : |z| = 2\}$

We give two examples of this.

### Example 3.1

Evaluate

$$\int_{\Gamma} \frac{ze^z}{(z-1)^2} dz,$$

where  $\Gamma$  is the circle  $\{z : |z| = 2\}$ .

### Solution

We use Cauchy's First Derivative Formula with  $f(z) = ze^z$ ,  $\alpha = 1$  and  $\mathcal{R} = \mathbb{C}$ . Then  $\mathcal{R}$  is simply connected,  $\Gamma$  is a simple-closed contour in  $\mathcal{R}$ , and  $\alpha$  lies inside  $\Gamma$  (see Figure 3.1). Also,  $f$  is analytic on  $\mathcal{R}$ .

It follows from Cauchy's First Derivative Formula that

$$\int_{\Gamma} \frac{ze^z}{(z-1)^2} dz = 2\pi i f'(1).$$

But  $f'(z) = e^z + ze^z$ , so  $f'(1) = 2e$ . Thus

$$\int_{\Gamma} \frac{ze^z}{(z-1)^2} dz = 4\pi ei.$$

In the next example we are required to find the partial fraction expansion of an expression that has a quadratic factor  $(z+2)^2$  in the denominator. To do this, observe that, more generally, a factor  $(z+\alpha)^n$  in the denominator of a rational function leads to the  $n$  terms

$$\frac{A_1}{z+\alpha}, \frac{A_2}{(z+\alpha)^2}, \dots, \frac{A_n}{(z+\alpha)^n}$$

in the partial fraction expansion of the rational function, some of which may be zero.

### Example 3.2

Evaluate

$$\int_{\Gamma} \frac{4 \cos z}{z(z+2)^2} dz,$$

where  $\Gamma$  is the circle  $\{z : |z+1| = 3\}$ .

### Solution

First we express  $4/(z(z+2)^2)$  in partial fractions:

$$\frac{4}{z(z+2)^2} = \frac{A}{z} + \frac{B}{z+2} + \frac{C}{(z+2)^2},$$

where  $A$ ,  $B$  and  $C$  are complex constants. Note that the repeated factor  $(z+2)^2$  in  $z(z+2)^2$  leads to the two terms  $B/(z+2)$

and  $C/(z+2)^2$ .

Multiplying both sides by  $z(z+2)^2$ , we obtain

$$\begin{aligned} 4 &= A(z+2)^2 + Bz(z+2) + Cz \\ &= A(z^2 + 4z + 4) + B(z^2 + 2z) + Cz. \end{aligned}$$

By equating the coefficients of  $z^2$ ,  $z$  and constants, we obtain

$$0 = A + B, \quad 0 = 4A + 2B + C \quad \text{and} \quad 4 = 4A,$$

which can be solved to give  $A = 1$ ,  $B = -1$ ,  $C = -2$ . Thus

$$\frac{4}{z(z+2)^2} = \frac{1}{z} - \frac{1}{z+2} - \frac{2}{(z+2)^2},$$

so

$$\int_{\Gamma} \frac{4 \cos z}{z(z+2)^2} dz = \int_{\Gamma} \frac{\cos z}{z} dz - \int_{\Gamma} \frac{\cos z}{z+2} dz - 2 \int_{\Gamma} \frac{\cos z}{(z+2)^2} dz.$$

We now use Cauchy's Integral Formula and First Derivative Formula with  $f(z) = \cos z$ ,  $\alpha = 0$  and  $-2$ , and  $\mathcal{R} = \mathbb{C}$ . Then  $\mathcal{R}$  is simply connected,  $\Gamma$  is a simple-closed contour in  $\mathcal{R}$ , and both values of  $\alpha$  lie inside  $\Gamma$  (see Figure 3.2). Also,  $f$  is analytic on  $\mathcal{R}$ .

It follows from Cauchy's Integral Formula that

$$\int_{\Gamma} \frac{\cos z}{z} dz = 2\pi i f(0) = 2\pi i \cos 0 = 2\pi i$$

and

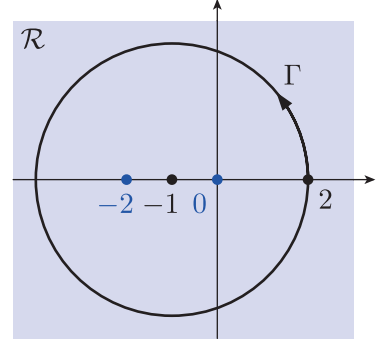
$$\int_{\Gamma} \frac{\cos z}{z+2} dz = 2\pi i f(-2) = 2\pi i \cos(-2) = 2\pi i \cos 2,$$

and it follows from Cauchy's First Derivative Formula that

$$\begin{aligned} \int_{\Gamma} \frac{\cos z}{(z+2)^2} dz &= 2\pi i f'(-2) \\ &= 2\pi i (-\sin(-2)) \\ &= 2\pi i \sin 2. \end{aligned}$$

Putting all this together, we obtain

$$\int_{\Gamma} \frac{4 \cos z}{z(z+2)^2} dz = 2\pi i (1 - \cos 2 - 2 \sin 2).$$



**Figure 3.2** The points 0 and  $-2$  inside the circle  $\Gamma = \{z : |z + 1| = 3\}$

### Exercise 3.1

Evaluate the following integrals, where  $\Gamma$  is the circle  $\{z : |z| = 3\}$ .

$$(a) \int_{\Gamma} \frac{e^{2z}}{(z+1)^2} dz \quad (b) \int_{\Gamma} \frac{e^{2z}}{z(z+1)^2} dz$$

We now present a proof of Cauchy's First Derivative Formula, which may be omitted on a first reading.

**Proof of Cauchy's First Derivative Formula** There are four steps in the proof.

1. Consider the integral

$$I = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)}{(z - \alpha)^2} dz.$$

By the Shrinking Contour Theorem (Theorem 1.4), we can replace  $\Gamma$  by a circle  $C$ , with centre  $\alpha$  and radius  $r$ , lying inside  $\Gamma$ . Thus

$$I = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - \alpha)^2} dz.$$

To prove that  $I = f'(\alpha)$ , we need to show that

$$\lim_{h \rightarrow 0} \frac{1}{h} (f(\alpha + h) - f(\alpha)) = I.$$

2. Since we are going to let  $h \rightarrow 0$ , we can assume that  $\alpha + h$  lies inside  $C$  (see Figure 3.3), so  $|h| < r$ . Thus, by Cauchy's Integral Formula, we have

$$f(\alpha) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - \alpha} dz \quad \text{and} \quad f(\alpha + h) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - (\alpha + h)} dz.$$

It follows that

$$\begin{aligned} & \frac{1}{h} (f(\alpha + h) - f(\alpha)) - I \\ &= \frac{1}{2\pi i} \int_C \left( \frac{1}{h} \left( \frac{1}{z - \alpha - h} - \frac{1}{z - \alpha} \right) - \frac{1}{(z - \alpha)^2} \right) f(z) dz \\ &= \frac{1}{2\pi i} \int_C \left( \frac{h}{(z - \alpha - h)(z - \alpha)^2} \right) f(z) dz, \end{aligned}$$

after simplification.

3. We now use the Estimation Theorem to give an upper estimate for the modulus of this last integral.

For all  $z$  on  $C$ , we have  $|z - \alpha| = r$ , so, by the backwards form of the Triangle Inequality,

$$|z - \alpha - h| \geq |z - \alpha| - |h| = r - |h|, \quad \text{for } z \in C.$$

Also, since  $f$  is continuous on  $\mathcal{R}$ , it follows from the Boundedness Theorem (Theorem 5.3 of Unit A3) that, for some number  $K$ ,

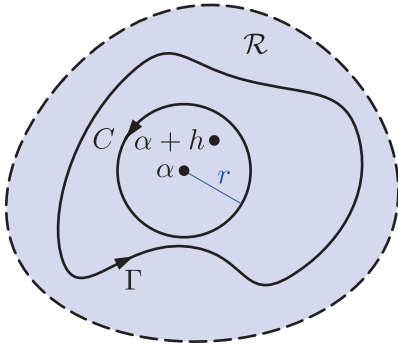
$$|f(z)| \leq K, \quad \text{for } z \in C.$$

Thus

$$\left| \frac{h}{(z - \alpha - h)(z - \alpha)^2} f(z) \right| \leq \frac{|h|K}{(r - |h|)r^2}, \quad \text{for } z \in C.$$

Since  $C$  has length  $2\pi r$ , we can apply the Estimation Theorem to give

$$\begin{aligned} \left| \frac{1}{2\pi i} \int_C \left( \frac{h}{(z - \alpha - h)(z - \alpha)^2} \right) f(z) dz \right| &\leq \frac{1}{2\pi} \times \frac{|h|K}{(r - |h|)r^2} \times 2\pi r \\ &= \frac{|h|K}{r(r - |h|)}. \end{aligned}$$



**Figure 3.3** Circle  $C$ , with centre  $\alpha$  and radius  $r$ , inside  $\Gamma$

4. From steps 2 and 3 we know that

$$\left| \frac{1}{h}(f(\alpha + h) - f(\alpha)) - I \right| \leq \frac{|h|K}{r(r - |h|)}.$$

If  $|h| < r/2$ , then  $r - |h| > r/2$ , so

$$\left| \frac{1}{h}(f(\alpha + h) - f(\alpha)) - I \right| < \frac{|h|K}{r \times r/2} = \frac{2K}{r^2}|h|.$$

Next we choose any positive number  $\varepsilon$ , and observe that if  $|h| < r/2$  and  $|h| < r^2\varepsilon/(2K)$ , then

$$\left| \frac{1}{h}(f(\alpha + h) - f(\alpha)) - I \right| < \frac{2K}{r^2} \times \frac{r^2\varepsilon}{2K} = \varepsilon.$$

Hence

$$\lim_{h \rightarrow 0} \frac{1}{h}(f(\alpha + h) - f(\alpha)) = I,$$

as required. ■

## 3.2 Cauchy's $n$ th Derivative Formula

We have just seen that, starting from Cauchy's Integral Formula for an analytic function, we can derive another contour integral which gives an expression for the first derivative  $f'$ . It is natural to ask whether we can repeat the procedure and derive an expression (involving an integral) for each of the higher derivatives, if these exist.

In order to discover such a formula, let us again abandon rigour temporarily, and differentiate each side of Cauchy's Integral Formula  $n$  times with respect to  $\alpha$  by using the formula

$$\begin{aligned} \frac{d^n}{d\alpha^n}((z - \alpha)^{-1}) &= \frac{d^{n-1}}{d\alpha^{n-1}}((z - \alpha)^{-2}) \\ &= \frac{d^{n-2}}{d\alpha^{n-2}}(2(z - \alpha)^{-3}) \\ &\vdots \\ &= n!(z - \alpha)^{-(n+1)}. \end{aligned}$$

Then

$$\begin{aligned} f^{(n)}(\alpha) &= \frac{d^n}{d\alpha^n} \left( \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)}{z - \alpha} dz \right) \\ &= \frac{1}{2\pi i} \int_{\Gamma} \frac{d^n}{d\alpha^n} \left( \frac{1}{z - \alpha} \right) f(z) dz \\ &= \frac{1}{2\pi i} \int_{\Gamma} \frac{n!}{(z - \alpha)^{n+1}} f(z) dz. \end{aligned}$$

This leads us to the formula

$$f^{(n)}(\alpha) = \frac{n!}{2\pi i} \int_{\Gamma} \frac{f(z)}{(z - \alpha)^{n+1}} dz.$$

A rigorous proof that this formula does indeed hold for  $n = 2, 3, \dots$  can be given along the same lines as the method for proving Cauchy's First Derivative Formula, using the Principle of Mathematical Induction. The details are more complicated, however, and in Unit B3 we will be able to deduce this formula from Taylor's Theorem. At this stage, therefore, we just state Cauchy's  $n$ th Derivative Formula and give some of its applications.

### Theorem 3.2 Cauchy's $n$ th Derivative Formula

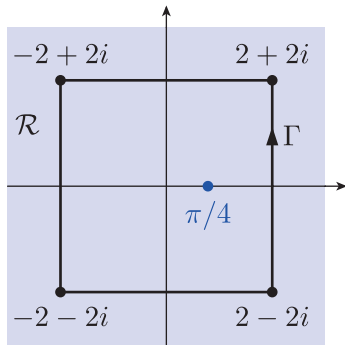
Let  $\mathcal{R}$  be a simply connected region, let  $\Gamma$  be a simple-closed contour in  $\mathcal{R}$ , and let  $f$  be a function that is analytic on  $\mathcal{R}$ . Then, for any point  $\alpha$  inside  $\Gamma$ ,  $f$  is  $n$ -times differentiable at  $\alpha$  and

$$f^{(n)}(\alpha) = \frac{n!}{2\pi i} \int_{\Gamma} \frac{f(z)}{(z - \alpha)^{n+1}} dz, \quad \text{for } n = 1, 2, \dots$$

When referencing Cauchy's  $n$ th Derivative Formula, we often simply refer to, for example, 'Cauchy's Second Derivative Formula' in order to specify the value of  $n$  (in this case  $n = 2$ ).

We can now evaluate integrals with higher powers of linear factors in their denominators, by writing Cauchy's  $n$ th Derivative Formula in the form

$$\int_{\Gamma} \frac{f(z)}{(z - \alpha)^{n+1}} dz = 2\pi i \frac{f^{(n)}(\alpha)}{n!}.$$



**Figure 3.4** The point  $\pi/4$  inside the square contour  $\Gamma$

### Example 3.3

Evaluate the integral

$$\int_{\Gamma} \frac{z \sin z}{(z - \pi/4)^3} dz,$$

where  $\Gamma$  is the square contour with vertices  $2 + 2i$ ,  $-2 + 2i$ ,  $-2 - 2i$  and  $2 - 2i$ .

### Solution

We use Cauchy's  $n$ th Derivative Formula with  $n = 2$ ,  $f(z) = z \sin z$ ,  $\alpha = \pi/4$  and  $\mathcal{R} = \mathbb{C}$ . Then  $\mathcal{R}$  is simply connected,  $\Gamma$  is a simple-closed contour in  $\mathcal{R}$ , and  $\alpha$  lies inside  $\Gamma$  (see Figure 3.4). Also,  $f$  is analytic on  $\mathcal{R}$ .

It follows from Cauchy's Second Derivative Formula that

$$\int_{\Gamma} \frac{z \sin z}{(z - \pi/4)^3} dz = 2\pi i \frac{f''(\pi/4)}{2!}.$$

But  $f'(z) = \sin z + z \cos z$ , and  $f''(z) = 2 \cos z - z \sin z$ , so  
 $f''(\pi/4) = 2 \cos(\pi/4) - (\pi/4) \sin(\pi/4) = \sqrt{2}(1 - \pi/8).$

Thus

$$\int_{\Gamma} \frac{z \sin z}{(z - \pi/4)^3} dz = \sqrt{2} \pi (1 - \pi/8) i.$$

### Exercise 3.2

Evaluate the following integrals, where  $\Gamma$  is the circle  $\{z : |z| = 2\}$ .

$$(a) \int_{\Gamma} \frac{\cosh 2z}{(z + i)^3} dz \quad (b) \int_{\Gamma} \frac{ze^z}{(z - 1)^{10}} dz \quad (c) \int_{\Gamma} \frac{e^{2z}}{z^3(z + 1)} dz$$

The method of evaluating integrals of the type in Example 3.3 and Exercise 3.2 is set out in a strategy given in Section 4.

The next exercise asks you to prove a result which will be needed later in the module.

### Exercise 3.3

Let  $\mathcal{R}$  be a simply connected region containing the circle

$$\Gamma = \{z : |z - \alpha| = r\}.$$

Let  $f$  be a function that is analytic on  $\mathcal{R}$ , and suppose that

$$|f(z)| \leq K, \quad \text{for } z \in \Gamma.$$

Use the Estimation Theorem and Cauchy's  $n$ th Derivative Formula to prove that, for any  $n \geq 1$ ,

$$|f^{(n)}(\alpha)| \leq Kn!/r^n.$$

The result from Exercise 3.3 is called **Cauchy's Estimate**.

Cauchy's  $n$ th Derivative Formula is evidently a useful tool for evaluating certain types of contour integrals. However, the greater significance of this result is that it shows that if  $f$  is analytic on a region  $\mathcal{R}$ , then  $f$  has derivatives of all orders at any point  $\alpha$  in  $\mathcal{R}$ .

### Theorem 3.3 Analyticity of Derivatives

Let  $\mathcal{R}$  be a region, and let  $f$  be a function that is analytic on  $\mathcal{R}$ . Then  $f$  possesses derivatives of all orders on  $\mathcal{R}$ , so  $f^{(1)}, f^{(2)}, f^{(3)}, \dots$  are all analytic on  $\mathcal{R}$ .

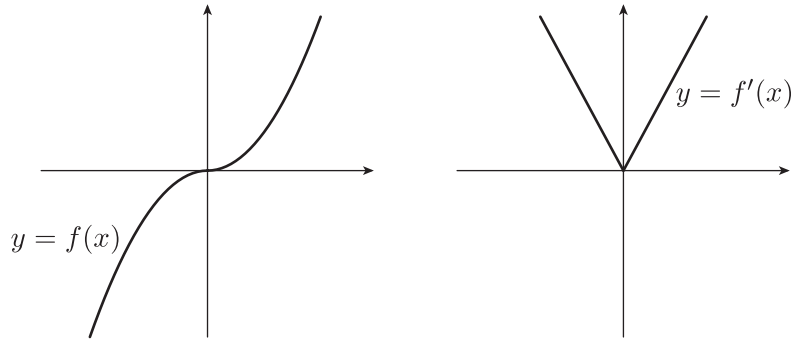
Note that  $\mathcal{R}$  is *not* assumed to be simply connected in this result.

**Proof** Let  $\alpha$  be any point of  $\mathcal{R}$ , let  $D$  be an open disc with centre  $\alpha$  lying in  $\mathcal{R}$ , and let  $\Gamma$  be any circle with centre  $\alpha$  lying in  $D$  – such a disc exists because  $\mathcal{R}$  is an open set. Then  $f^{(n)}(\alpha)$  exists, for  $n = 1, 2, \dots$ , and is given by Cauchy's  $n$ th Derivative Formula. ■

Note that the corresponding assertion for real functions does not hold. For example, consider the differentiable function

$$f(x) = \begin{cases} x^2, & x \geq 0, \\ -x^2, & x < 0. \end{cases}$$

Then  $f'(x) = 2|x|$ , so  $f'$  is not differentiable at 0 (see Figure 3.5).



**Figure 3.5** Graphs of  $y = f(x)$  and  $y = f'(x)$

### Exercise 3.4

Use the Analyticity of Derivatives to prove the following facts.

- (a) There is no analytic function  $F$  such that

$$F'(z) = |z|, \quad \text{for all } z \in \mathbb{C}.$$

- (b) If  $f$  is an entire function such that  $f'$  is bounded, then  $f$  is a linear function; that is,

$$f(z) = \alpha z + \beta, \quad \text{where } \alpha, \beta \in \mathbb{C}.$$

## Further exercises

### Exercise 3.5

Evaluate the following integrals by applying Cauchy's  $n$ th Derivative Formula. In each case,  $\Gamma = \{z : |z| = 2\}$ .

- (a)  $\int_{\Gamma} \frac{\cos z}{(z - \pi/2)^2} dz$     (b)  $\int_{\Gamma} \frac{\cosh \pi z}{(z - i)^3} dz$     (c)  $\int_{\Gamma} \frac{\sin z}{(z^2 + 2z - 3)^2} dz$   
 (d)  $\int_{\Gamma} \frac{\sin 2z}{(z - \pi/4)^5} dz$     (e)  $\int_{\Gamma} \frac{1}{(z + 1)^{11}} dz$

**Exercise 3.6**

Evaluate

$$\int_{\Gamma} \frac{e^{3z}}{z^4 - 2z^3 + z^2} dz,$$

where  $\Gamma = \{z : |z| = 3\}$ .

## 4 Revision of contour integration

After working through this section, you should be able to:

- identify and use appropriate techniques for evaluating various contour integrals.

In this unit and the previous one you have met a number of methods for evaluating complex integrals. In this section we will review some of these methods. Let us begin by listing the techniques that we will use.

### Parametrisation

Let  $\Gamma : \gamma(t)$  ( $t \in [a, b]$ ) be a smooth path in  $\mathbb{C}$ , and let  $f$  be a function that is continuous on  $\Gamma$ . Then, by definition,

$$\int_{\Gamma} f(z) dz = \int_a^b f(\gamma(t)) \gamma'(t) dt.$$

The integral of  $f$  along a contour is the sum of the integrals of  $f$  along its constituent smooth paths. (See Subsection 2.2 of Unit B1.)

### Closed Contour Theorem

Let  $f$  be a function that is continuous and has a primitive  $F$  on a region  $\mathcal{R}$ . Then

$$\int_{\Gamma} f(z) dz = 0,$$

for any closed contour  $\Gamma$  in  $\mathcal{R}$ . (See Theorem 3.4 of Unit B1.)

### Cauchy's Theorem

Let  $\mathcal{R}$  be a simply connected region, and let  $f$  be a function that is analytic on  $\mathcal{R}$ . Then

$$\int_{\Gamma} f(z) dz = 0,$$

for any closed contour  $\Gamma$  in  $\mathcal{R}$ . (See Theorem 1.2.)

### Cauchy's Integral Formula

Let  $\mathcal{R}$  be a simply connected region, let  $\Gamma$  be a simple-closed contour in  $\mathcal{R}$ , and let  $f$  be a function that is analytic on  $\mathcal{R}$ . Then

$$f(\alpha) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)}{z - \alpha} dz,$$

for any point  $\alpha$  inside  $\Gamma$ . (See Theorem 2.1.)

### Cauchy's $n$ th Derivative Formula

Let  $\mathcal{R}$  be a simply connected region, let  $\Gamma$  be a simple-closed contour in  $\mathcal{R}$ , and let  $f$  be a function that is analytic on  $\mathcal{R}$ . Then, for any point  $\alpha$  inside  $\Gamma$ ,  $f$  is  $n$ -times differentiable at  $\alpha$  and

$$f^{(n)}(\alpha) = \frac{n!}{2\pi i} \int_{\Gamma} \frac{f(z)}{(z - \alpha)^{n+1}} dz, \quad \text{for } n = 1, 2, \dots$$

(See Theorem 3.2.)

The results of the following exercise in partial fractions are needed later.

#### Exercise 4.1

Expand each of the following expressions as partial fractions.

(a)  $\frac{1}{z(z-3)}$       (b)  $\frac{1}{z^2(z-3)}$       (c)  $\frac{1}{z^3(z-3)}$

We now look at several examples to demonstrate the different techniques for evaluating integrals. In these examples, we continue to use our convention that if a parametrisation of a simple-closed contour is not specified, then it is assumed that the contour is traversed once anticlockwise.

#### Example 4.1

Evaluate

$$\int_{\Gamma} \frac{1}{(z-i)^2} dz,$$

where  $\Gamma$  is the circle  $\{z : |z| = 2\}$ .

#### Solution

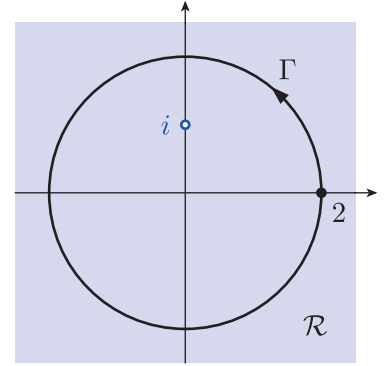
We could try the method of parametrisation; however, that method leads to an awkward algebraic expression that is difficult to manipulate. As an alternative, we might consider applying Cauchy's Theorem. But the integrand  $f(z) = 1/(z-i)^2$  is not analytic at the point  $i$ , which lies inside  $\Gamma$ , so there is no *simply connected* region containing  $\Gamma$  on which  $f$  is analytic.

Instead we apply the Closed Contour Theorem, which still requires us to find a region  $\mathcal{R}$  that contains  $\Gamma$ , but that region need not be simply connected. We choose  $\mathcal{R} = \mathbb{C} - \{i\}$ , which contains the contour  $\Gamma$ , as shown in Figure 4.1. The function  $f$  is continuous on  $\mathcal{R}$  and has a primitive

$$F(z) = -\frac{1}{z-i}$$

on  $\mathcal{R}$ . Hence, by the Closed Contour Theorem,

$$\int_{\Gamma} \frac{1}{(z-i)^2} dz = 0.$$



**Figure 4.1** The point  $i$  inside the circle  $\Gamma = \{z : |z| = 2\}$

Let us consider another example.

### Example 4.2

Evaluate

$$\int_{\Gamma} \frac{1}{z - 1/2} dz,$$

where  $\Gamma$  is the unit circle  $\{z : |z| = 1\}$ .

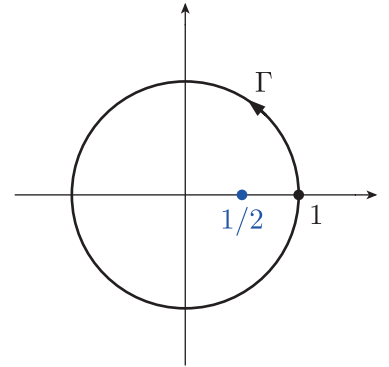
### Solution

The method of parametrisation again leads to an awkward algebraic expression. We cannot use Cauchy's Theorem either, because the integrand  $1/(z - 1/2)$  is not analytic at the point  $1/2$ , which lies inside the unit circle, as illustrated in Figure 4.2.

Next we might consider the Closed Contour Theorem, perhaps making use of the function  $F(z) = \text{Log}(z - 1/2)$ . This function satisfies  $F'(z) = 1/(z - 1/2)$ , for  $z \in \mathbb{C} - \{x \in \mathbb{R} : x \leq 1/2\}$ . However,  $F$  is *not* a primitive of  $1/(z - 1/2)$  on any region that contains a contour such as  $\Gamma$  that encircles the point  $1/2$  (because  $F$  is not continuous on such a contour, let alone differentiable).

Instead we use Cauchy's Integral Formula with  $\mathcal{R} = \mathbb{C}$  and  $f(z) = 1$ . Then  $\alpha = 1/2$  lies inside the simple-closed contour  $\Gamma$ , which is contained in  $\mathcal{R}$ , and  $f$  is analytic on  $\mathcal{R}$ . So we can apply Cauchy's Integral Formula to see that

$$\int_{\Gamma} \frac{1}{z - 1/2} dz = 2\pi i f(1/2) = 2\pi i.$$



**Figure 4.2** The point  $1/2$  inside the circle  $\Gamma = \{z : |z| = 1\}$

A similar argument shows that if  $\Gamma$  is any circle, and  $\alpha$  is a point *inside*  $\Gamma$ , then the integral of  $f(z) = 1/(z - \alpha)$  around  $\Gamma$  is  $2\pi i$ . On the other hand, if  $\alpha$  lies *outside*  $\Gamma$ , then the integral of  $f$  around  $\Gamma$  is 0, by Exercise 1.4.

In summary,

$$\int_{\Gamma} \frac{1}{z - \alpha} dz = \begin{cases} 2\pi i, & \text{if } \alpha \text{ lies inside } \Gamma, \\ 0, & \text{if } \alpha \text{ lies outside } \Gamma. \end{cases} \quad (4.1)$$

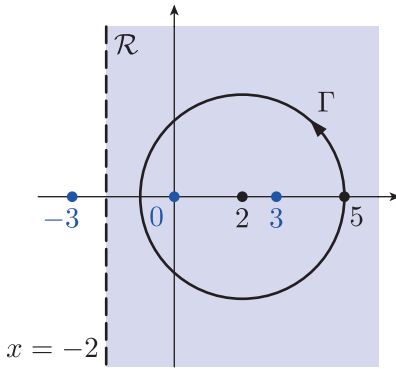
Here is an exercise to try for yourself.

### Exercise 4.2

Evaluate the following integrals, where  $\Gamma$  is the circle  $\{z : |z - 1| = 2\}$ .

$$(a) \int_{\Gamma} \frac{1}{z} dz \quad (b) \int_{\Gamma} \frac{1}{z - 5} dz \quad (c) \int_{\Gamma} \frac{1}{(z - 1)^2} dz$$

The next example combines Cauchy's Integral Formula with the use of partial fractions.



**Figure 4.3** The points 0 and 3 inside the circle  $\Gamma = \{z : |z - 2| = 3\}$

### Example 4.3

Evaluate

$$\int_{\Gamma} \frac{e^z}{z(z^2 - 9)} dz,$$

where  $\Gamma$  is the circle  $\{z : |z - 2| = 3\}$ .

### Solution

The integrand is analytic except at the points 0, 3 and  $-3$ . The first two of these lie inside  $\Gamma$ , and the last one lies outside  $\Gamma$ , as illustrated in Figure 4.3. To apply Cauchy's Integral Formula, we must choose a simply connected region  $\mathcal{R}$  that includes the circle  $\Gamma$  but omits the point  $-3$ ; the region  $\mathcal{R} = \{z : \operatorname{Re} z > -2\}$  will do.

We choose  $f(z) = e^z/(z + 3)$ , which is analytic on  $\mathcal{R}$ . (We could have chosen  $f(z) = e^z$  instead, but that choice would make the calculations more complicated.) The integrand can now be written as

$$\frac{f(z)}{z(z - 3)},$$

but this is not in a suitable form to apply Cauchy's Integral Formula because the denominator has two linear factors rather than one. To get round this problem, we split up the denominator using partial fractions, as follows:

$$\frac{1}{z(z - 3)} = \frac{1/3}{z - 3} - \frac{1/3}{z}$$

(by Exercise 4.1(a)). Therefore

$$\int_{\Gamma} \frac{e^z}{z(z^2 - 9)} dz = \frac{1}{3} \int_{\Gamma} \frac{f(z)}{z - 3} dz - \frac{1}{3} \int_{\Gamma} \frac{f(z)}{z} dz.$$

We now apply Cauchy's Integral Formula to each of the two integrals on the right, where  $f(z) = e^z/(z+3)$ , to obtain

$$\begin{aligned}\int_{\Gamma} \frac{e^z}{z(z^2-9)} dz &= \frac{1}{3} \times 2\pi i f(3) - \frac{1}{3} \times 2\pi i f(0) \\ &= \frac{2\pi i}{3} \left( \frac{e^3}{6} - \frac{e^0}{3} \right) \\ &= \frac{\pi}{9} (e^3 - 2)i.\end{aligned}$$

When the denominator of the integrand contains factors  $(z - \alpha)^n$ , where  $n > 1$ , then we can apply a similar strategy to that of Example 4.3, but using Cauchy's  $n$ th Derivative Formula as well as Cauchy's Integral Formula.

### Example 4.4

Evaluate

$$\int_{\Gamma} \frac{e^z}{z^2(z^2-9)} dz,$$

where  $\Gamma$  is the circle  $\{z : |z - 2| = 3\}$ .

### Solution

This integral is similar to that of Example 4.3, except that the  $z$  term in the denominator of the integrand from the earlier example has been replaced by  $z^2$ .

We use the same simply connected region  $\mathcal{R}$  as before, and the same function  $f(z) = e^z/(z+3)$ . This time, however, we need the partial fraction expansion of  $1/(z^2(z-3))$ , which is

$$\frac{1}{z^2(z-3)} = \frac{1/9}{z-3} - \frac{1/9}{z} - \frac{1/3}{z^2},$$

by Exercise 4.1(b).

It follows that

$$\int_{\Gamma} \frac{f(z)}{z^2(z-3)} dz = \frac{1}{9} \int_{\Gamma} \frac{f(z)}{z-3} dz - \frac{1}{9} \int_{\Gamma} \frac{f(z)}{z} dz - \frac{1}{3} \int_{\Gamma} \frac{f(z)}{z^2} dz.$$

The first two integrals on the right can be evaluated using Cauchy's Integral Formula, and the third integral can be evaluated using Cauchy's First Derivative Formula. For the third integral we need the derivative of  $f$ , which is

$$f'(z) = \frac{(z+3)e^z - 1 \times e^z}{(z+3)^2} = \frac{(z+2)e^z}{(z+3)^2}.$$

We obtain

$$\begin{aligned}\int_{\Gamma} \frac{f(z)}{z^2(z-3)} dz &= \frac{1}{9} \times 2\pi i f(3) - \frac{1}{9} \times 2\pi i f(0) - \frac{1}{3} \times 2\pi i f'(0) \\ &= \frac{2\pi i}{9} \left( \frac{e^3}{6} - \frac{e^0}{3} - 3 \times \frac{2e^0}{9} \right) \\ &= \frac{\pi}{27} (e^3 - 6)i.\end{aligned}$$

Here is a similar integral for you to try.

### Exercise 4.3

Evaluate

$$\int_{\Gamma} \frac{e^z}{z^3(z^2-9)} dz,$$

where  $\Gamma$  is the circle  $\{z : |z-2| = 3\}$ .

By looking at these examples, we can extract a strategy for evaluating a wide variety of integrals.

### Strategy for evaluating contour integrals

To evaluate the integral

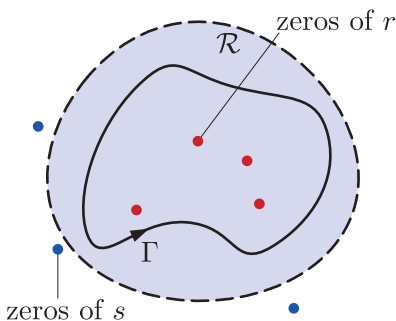
$$\int_{\Gamma} \frac{g(z)}{p(z)} dz,$$

where

- $\Gamma$  is a simple-closed contour
- $g$  is analytic on a simply connected region containing  $\Gamma$
- $p$  is a polynomial function with no zeros on  $\Gamma$ ,

carry out the following steps.

1. Factorise  $p(z)$  as  $r(z)s(z)$ , where the zeros of  $r$  lie inside  $\Gamma$  and the zeros of  $s$  lie outside  $\Gamma$ . Then the function  $f = g/s$  is analytic on a simply connected region  $\mathcal{R}$  that contains  $\Gamma$  but does not contain the zeros of  $s$ .
2. Expand  $1/r(z)$  in partial fractions.
3. Expand  $\int_{\Gamma} \frac{f(z)}{r(z)} dz$  as a sum of integrals that can be evaluated using Cauchy's Integral and  $n$ th Derivative Formulas.



**Figure 4.4** Zeros of  $r$  and  $s$

Some features of the strategy are illustrated in Figure 4.4.

### Exercise 4.4

Use the strategy to evaluate the following integrals, where

$$\Gamma = \{z : |z - i/2| = 3/4\}.$$

$$(a) \int_{\Gamma} \frac{e^{2z}}{z(z^2 + 1)} dz \quad (b) \int_{\Gamma} \frac{e^{2z}}{z^2(z^2 + 1)} dz$$

In Unit C1 you will learn about a powerful result called the Residue Theorem, which provides an alternative procedure for evaluating contour integrals to that presented in the strategy above.

## Further exercises

### Exercise 4.5

All but two of the following integrals can be evaluated by the methods introduced in this unit. Identify these two integrals and say how you would evaluate them. Evaluate the other seven integrals.

The contours  $\Gamma_1$ ,  $\Gamma_2$  and  $\Gamma_3$  are as follows:

$\Gamma_1$  is the triangular contour shown in Figure 4.5

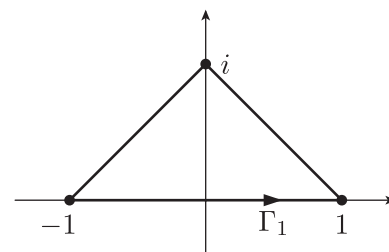
$\Gamma_2$  is the line segment from  $-1$  to  $i$

$\Gamma_3$  is the circle  $\{z : |z| = 2\}$ .

$$(a) \int_{\Gamma_1} \sin z \, dz \quad (b) \int_{\Gamma_1} \sin(z^2) \, dz \quad (c) \int_{\Gamma_1} \frac{1}{z+i} \, dz$$

$$(d) \int_{\Gamma_2} z \sin z \, dz \quad (e) \int_{\Gamma_3} \frac{\sin z}{z^2} \, dz \quad (f) \int_{\Gamma_3} \frac{\cosh z}{z^5} \, dz$$

$$(g) \int_{\Gamma_3} \frac{z^5}{\cosh \frac{1}{2}z} \, dz \quad (h) \int_{\Gamma_1} \frac{z}{4z^2 + 1} \, dz \quad (i) \int_{\Gamma_3} \operatorname{Re} z \, dz$$



**Figure 4.5** A triangular contour

## 5 Proof of Cauchy's Theorem

After working through this section, you should be able to:

- understand the proof of Cauchy's Theorem
- state Morera's Theorem.

In Subsection 1.2 we outlined a proof of Cauchy's Theorem. We now expand this outline to a full proof. The details of the proof are challenging, so you may choose to omit them on a first reading. However, make sure you get to Morera's Theorem at the end of the section, as it is needed later in the module.

A key ingredient in the proof is the Nested Rectangles Theorem (Theorem 5.6 of Unit A3). We state it here for convenience.

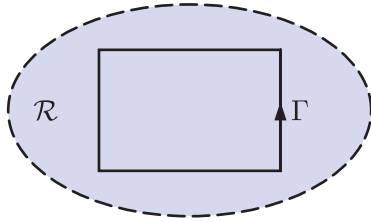
### Nested Rectangles Theorem

Let  $R_0, R_1, R_2, \dots$  be a sequence of closed rectangles with sides parallel to the axes, and with diagonals of lengths  $s_0, s_1, s_2, \dots$ , such that

$$R_0 \supseteq R_1 \supseteq R_2 \supseteq \dots \quad \text{and} \quad \lim_{n \rightarrow \infty} s_n = 0.$$

Then there is a unique complex number  $\alpha$  that lies in all of the rectangles  $R_n$ . Moreover, for each positive number  $\varepsilon$ , there is an integer  $N$  such that

$$R_n \subseteq \{z : |z - \alpha| < \varepsilon\}, \quad \text{for all } n > N.$$



**Figure 5.1** A rectangular contour

## 5.1 Cauchy's Theorem for rectangular contours

In this subsection we give the first stage of the proof of Cauchy's Theorem – the special case when  $\Gamma$  is a rectangular contour in  $\mathcal{R}$  (see Figure 5.1). This special case is in fact due to Édouard Goursat (1858–1936), a French mathematician who made contributions to refining Cauchy's Theorem. Earlier proofs of Cauchy's Theorem had made the unnecessary assumption that the derivative of  $f$  is a continuous function.

### Theorem 5.1 Cauchy's Theorem for rectangular contours

Let  $f$  be a function that is analytic on a simply connected region  $\mathcal{R}$ . Then

$$\int_{\Gamma} f(z) dz = 0,$$

for any rectangular contour  $\Gamma$  in  $\mathcal{R}$ .

**Proof** There are four steps in this proof.

Let

$$I = \int_{\Gamma} f(z) dz,$$

where  $\Gamma$  is a rectangular contour in  $\mathcal{R}$ .

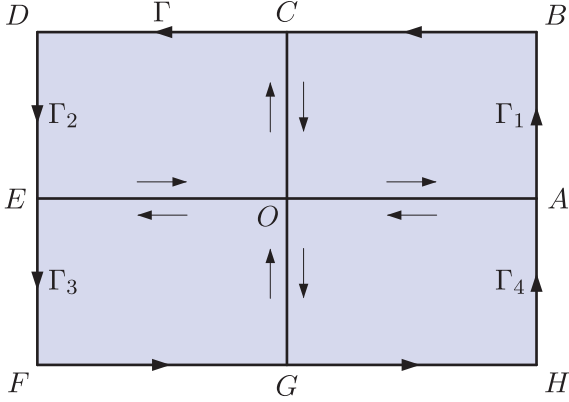
1. We first construct a sequence  $\Delta_0, \Delta_1, \Delta_2, \dots$  of rectangular contours in  $\mathcal{R}$  such that

$$|I| \leq 4^n \left| \int_{\Delta_n} f(z) dz \right|, \quad \text{for } n = 0, 1, 2, \dots$$

We use the Greek capital letter  $\Delta$  (delta) for these contours because  $\Gamma$  is already in use.

To begin, let  $\Delta_0 = \Gamma$ .

In order to construct  $\Delta_1$ , we split the interior of  $\Gamma$  into four congruent rectangles with boundary contours  $\Gamma_1, \Gamma_2, \Gamma_3$  and  $\Gamma_4$ , the vertices of which are labelled  $O, A, B, C, D, E, F, G$  and  $H$  (see Figure 5.2).



**Figure 5.2**  $\Gamma_1 : OABCO$ ,  $\Gamma_2 : OCDEO$ ,  $\Gamma_3 : OEFGO$ ,  $\Gamma_4 : OGHAO$

Consider the sum

$$\int_{\Gamma_1} f(z) dz + \int_{\Gamma_2} f(z) dz + \int_{\Gamma_3} f(z) dz + \int_{\Gamma_4} f(z) dz.$$

Let us denote the line segment from  $O$  to  $A$  by  $OA$  (and similarly for other line segments in Figure 5.2). Since

$$\begin{aligned} \int_{OA} f(z) dz &= - \int_{AO} f(z) dz, & \int_{OC} f(z) dz &= - \int_{CO} f(z) dz, \\ \int_{OE} f(z) dz &= - \int_{EO} f(z) dz, & \int_{OG} f(z) dz &= - \int_{GO} f(z) dz, \end{aligned}$$

we have

$$\int_{\Gamma_1} f(z) dz + \int_{\Gamma_2} f(z) dz + \int_{\Gamma_3} f(z) dz + \int_{\Gamma_4} f(z) dz = I,$$

so, by the Triangle Inequality,

$$|I| \leq \left| \int_{\Gamma_1} f(z) dz \right| + \left| \int_{\Gamma_2} f(z) dz \right| + \left| \int_{\Gamma_3} f(z) dz \right| + \left| \int_{\Gamma_4} f(z) dz \right|.$$

Now let  $k$  be such that

$$\left| \int_{\Gamma_k} f(z) dz \right|$$

is the largest of the four terms on the right (the choice of  $k$  may not be unique), and let  $\Delta_1 = \Gamma_k$ . Then

$$|I| \leq 4 \left| \int_{\Delta_1} f(z) dz \right|.$$

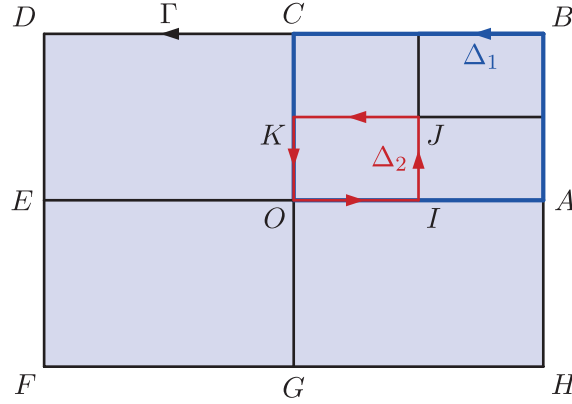
Now suppose that the original contour  $\Gamma$  has length  $l$ . Then the length of the new contour  $\Delta_1$  is  $\frac{1}{2}l$ ; we write  $L(\Delta_1) = \frac{1}{2}l$ .

In summary, we have chosen  $\Delta_1$  such that

$$|I| \leq 4 \left| \int_{\Delta_1} f(z) dz \right| \quad \text{and} \quad L(\Delta_1) = \frac{1}{2}l. \quad (5.1)$$

We now repeat the argument above starting with  $\Delta_1$  instead of  $\Gamma$ ; we split the interior of  $\Delta_1$  into four congruent rectangles, and let  $\Delta_2$  be the boundary of one of these, chosen so that the integral around it has the largest modulus: Figure 5.3 shows an example of this, in the case that  $\Delta_1 = \Gamma_1$ . Hence, as before,

$$\left| \int_{\Delta_1} f(z) dz \right| \leq 4 \left| \int_{\Delta_2} f(z) dz \right|.$$



**Figure 5.3**  $\Gamma : BDFHB$ ,  $\Delta_1 : OABCO$ ,  $\Delta_2 : OIJKO$

Also, the length of  $\Delta_2$  is half of the length of  $\Delta_1$ ; that is,  $L(\Delta_2) = \frac{1}{2}L(\Delta_1)$ . Combining these results with those of equation (5.1), we obtain

$$|I| \leq 4^2 \left| \int_{\Delta_2} f(z) dz \right| \quad \text{and} \quad L(\Delta_2) = \frac{1}{4}l.$$

We now repeat the argument indefinitely to produce a sequence of nested rectangular contours  $\Delta_n$ ,  $n = 0, 1, 2, \dots$ , such that

$$|I| \leq 4^n \left| \int_{\Delta_n} f(z) dz \right| \quad (5.2)$$

and

$$L(\Delta_n) = l/2^n. \quad (5.3)$$

2. Next we use the Nested Rectangles Theorem. For  $n = 0, 1, 2, \dots$ , let  $R_n$  be the closed rectangle consisting of  $\Delta_n$  and its inside. Then

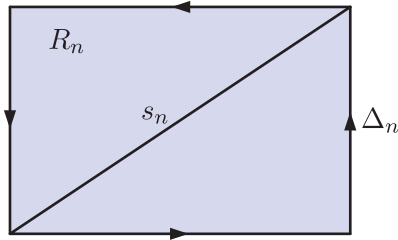
$$R_0 \supseteq R_1 \supseteq R_2 \supseteq \dots$$

Also, for each rectangle  $R_n$ , the length  $s_n$  of its diagonal (see Figure 5.4) satisfies

$$0 \leq s_n \leq L(\Delta_n).$$

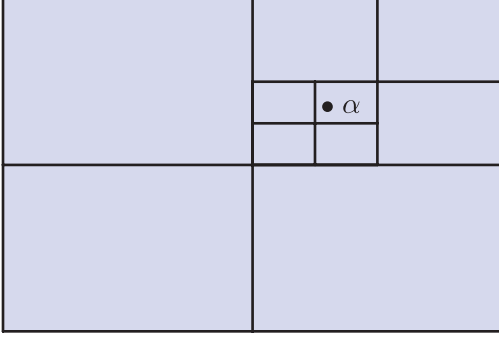
Since  $\lim_{n \rightarrow \infty} L(\Delta_n) = \lim_{n \rightarrow \infty} l/2^n = 0$ , it follows from the Squeeze Rule for sequences (Theorem 1.1 of Unit A3) that

$$\lim_{n \rightarrow \infty} s_n = 0.$$



**Figure 5.4** The rectangle  $R_n$

The Nested Rectangles Theorem then tells us that there is a unique complex number  $\alpha$  that lies in all of the rectangles  $R_n$  (see Figure 5.5).



**Figure 5.5** Complex number  $\alpha$  that lies in all of the rectangles  $R_n$

3. Since  $f$  is an analytic function, it is differentiable at  $\alpha$ . Hence, by the Linear Approximation Theorem (Theorem 1.2 of Unit A4), we can write

$$f(z) = f(\alpha) + (z - \alpha)f'(\alpha) + e(z),$$

where  $e(z)/(z - \alpha) \rightarrow 0$  as  $z \rightarrow \alpha$ . The function  $e$  is analytic on  $\mathcal{R}$ , since  $f$  is analytic on  $\mathcal{R}$ . Let us now define

$$r(z) = \frac{e(z)}{z - \alpha} \quad (z \in \mathcal{R} - \{\alpha\}),$$

which is analytic and hence continuous on  $\mathcal{R} - \{\alpha\}$ . Then

$$f(z) = f(\alpha) + (z - \alpha)f'(\alpha) + (z - \alpha)r(z),$$

where  $r(z) \rightarrow 0$  as  $z \rightarrow \alpha$ . We now extend the domain of  $r$  to  $\mathcal{R}$  by defining  $r(\alpha) = 0$ . Since  $\alpha$  is a limit point of  $\mathcal{R} - \{\alpha\}$ , and

$$\lim_{z \rightarrow \alpha} r(z) = r(\alpha) = 0,$$

it follows from Theorem 3.1 of Unit A3 that  $r$  is continuous at  $\alpha$ . So  $r$  is continuous throughout  $\mathcal{R}$ .

Integrating around  $\Delta_n$ , we obtain

$$\int_{\Delta_n} f(z) dz = \int_{\Delta_n} (f(\alpha) + (z - \alpha)f'(\alpha)) dz + \int_{\Delta_n} (z - \alpha)r(z) dz.$$

The first integral on the right-hand side is 0 since the integrand is a polynomial, and the Closed Contour Theorem can be applied. Thus

$$\int_{\Delta_n} f(z) dz = \int_{\Delta_n} (z - \alpha)r(z) dz.$$

4. Finally, we apply the Estimation Theorem (Theorem 4.1 of Unit B1) to the previous integral.

By equation (5.3), the length of the contour is  $L(\Delta_n) = l/2^n$ .

Now suppose that  $\varepsilon > 0$  is given. Since  $r(z) \rightarrow 0$  as  $z \rightarrow \alpha$ , we know that there is a  $\delta > 0$  such that

$$|z - \alpha| < \delta \implies |r(z)| < \varepsilon.$$

By the Nested Rectangles Theorem, we can choose an integer  $N$  in such a way that, for  $n > N$ , the contour  $\Delta_n$  lies entirely within the disc  $\{z : |z - \alpha| < \delta\}$ . Thus, for  $z \in \Delta_n$ , where  $n > N$ , we have

$$|(z - \alpha)r(z)| = |z - \alpha||r(z)| < (l/2^n)\varepsilon,$$

since  $|z - \alpha| < s_n$  and  $s_n \leq L(\Delta_n) = l/2^n$ . It follows, on applying the Estimation Theorem with  $M = (l/2^n)\varepsilon$  and  $L(\Delta_n) = l/2^n$ , that

$$\left| \int_{\Delta_n} f(z) dz \right| = \left| \int_{\Delta_n} (z - \alpha)r(z) dz \right| \leq \frac{l\varepsilon}{2^n} \times \frac{l}{2^n} = \frac{l^2\varepsilon}{4^n}.$$

Hence, by inequality (5.2),

$$|I| \leq 4^n \left| \int_{\Delta_n} f(z) dz \right| \leq 4^n \times \frac{l^2\varepsilon}{4^n} = l^2\varepsilon.$$

Since  $|I| \leq l^2\varepsilon$  for each positive  $\varepsilon$ , it follows that  $I = 0$  (using Remark 1 after Theorem 2.1 of Subsection 2.2). ■

## 5.2 Cauchy's Theorem for closed grid paths

The next stage in the proof of Cauchy's Theorem is to prove the theorem for the case of a closed grid path. Recall from Subsection 3.2 of Unit B1 that a grid path is a contour made up of line segments that are parallel to the  $x$ - and  $y$ -axes.

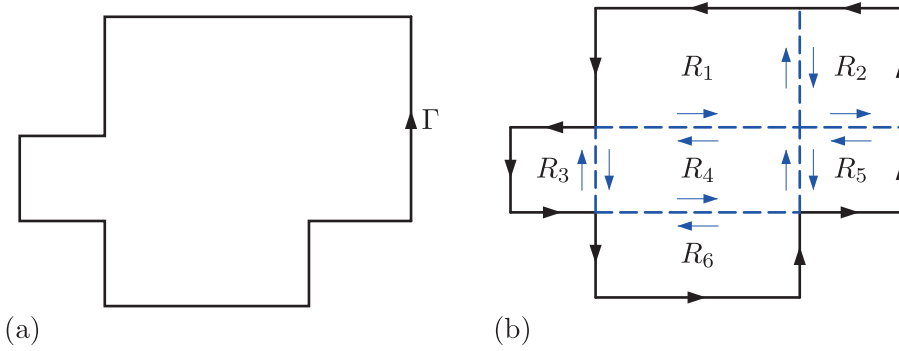
### Theorem 5.2 Cauchy's Theorem for closed grid paths

Let  $f$  be a function that is analytic on a simply connected region  $\mathcal{R}$ . Then

$$\int_{\Gamma} f(z) dz = 0,$$

for any closed grid path  $\Gamma$  in  $\mathcal{R}$ .

**Proof** Suppose first that  $\Gamma$  is a *simple-closed* grid path, such as the one illustrated in Figure 5.6(a). Then we can introduce extra horizontal and vertical line segments, as shown in Figure 5.6(b), to divide the inside of  $\Gamma$  into rectangles  $R_1, R_2, \dots, R_n$ , whose interiors are disjoint; for example, this can be done by extending to the inside of  $\Gamma$  all the horizontal and vertical line segments in  $\Gamma$ . (There are other methods of dividing the inside of  $\Gamma$  into rectangles; it does not matter which method we pick.)



**Figure 5.6** Dividing the inside of a simple-closed grid path

Now note that the integral of  $f$  around each of the boundaries  $\Gamma_1, \Gamma_2, \dots, \Gamma_n$  of the rectangles  $R_1, R_2, \dots, R_n$  is zero, by Cauchy's Theorem for rectangular contours (Theorem 5.1). So

$$\int_{\Gamma_1} f + \int_{\Gamma_2} f + \dots + \int_{\Gamma_n} f = 0.$$

Also, as in the proof of Theorem 5.1, the integral of  $f$  along each of the line segments inside  $\Gamma$  occurs twice in this sum of integrals (once in each direction), so they cancel in pairs. It follows that

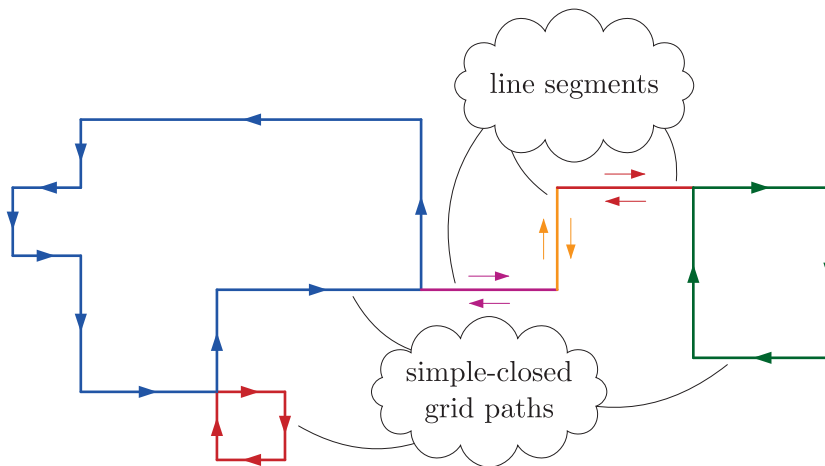
$$\int_{\Gamma} f = \int_{\Gamma_1} f + \int_{\Gamma_2} f + \dots + \int_{\Gamma_n} f = 0,$$

as required.

To complete the proof we note that any *closed* grid path can be decomposed into:

- a finite number of simple-closed grid paths, and
- a finite number of line segments, each traversed in both directions.

This fact is illustrated in Figure 5.7.



**Figure 5.7** A 'decomposed' closed grid path

Since the integral of  $f$  around a simple-closed grid path is zero, and the integrals of  $f$  along the line segments cancel in pairs, we deduce that the integral of  $f$  along any closed grid path is zero. ■

## 5.3 Primitive Theorem

In this subsection we present the final stage in the proof of Cauchy's Theorem. We start by proving the Primitive Theorem.

### Theorem 5.3 Primitive Theorem

Let  $f$  be a function that is analytic on a simply connected region  $\mathcal{R}$ . Then  $f$  has a primitive on  $\mathcal{R}$ .

The Primitive Theorem is also known as the Antiderivative Theorem. We prove it as a consequence of the following lemma.

### Lemma 5.1

Let  $f$  be a function that is continuous on a region  $\mathcal{R}$  and satisfies

$$\int_{\Gamma} f(z) dz = 0,$$

for all closed grid paths  $\Gamma$  in  $\mathcal{R}$ . Then  $f$  has a primitive on  $\mathcal{R}$ .

The Primitive Theorem is an immediate consequence of this lemma and Cauchy's Theorem for closed grid paths (Theorem 5.2). This is because the hypotheses of the Primitive Theorem are that  $f$  is analytic on a *simply connected* region  $\mathcal{R}$ , so we can apply Theorem 5.2 to see that

$$\int_{\Gamma} f(z) dz = 0,$$

for all closed grid paths  $\Gamma$  in  $\mathcal{R}$ . Hence  $f$  has a primitive on  $\mathcal{R}$ , by Lemma 5.1.

It remains to prove the lemma.

**Proof of Lemma 5.1** First we introduce the notation

$$\int_{\alpha}^{\beta} f(z) dz, \quad \text{for } \alpha, \beta \in \mathcal{R},$$

to denote the integral of  $f$  along any grid path in  $\mathcal{R}$  from  $\alpha$  to  $\beta$ ; such a path exists by the Grid Path Theorem (Theorem 3.5 of Unit B1). There is no ambiguity in this definition because the value of the integral does not depend on which grid path from  $\alpha$  to  $\beta$  we choose. (This is because if  $\Gamma_1$  and  $\Gamma_2$  are both grid paths from  $\alpha$  to  $\beta$ , then

$$\begin{aligned} \int_{\Gamma_1} f(z) dz - \int_{\Gamma_2} f(z) dz &= \int_{\Gamma_1} f(z) dz + \int_{\tilde{\Gamma}_2} f(z) dz, \\ &= \int_{\Gamma_1 + \tilde{\Gamma}_2} f(z) dz = 0, \end{aligned}$$

since  $\Gamma_1 + \tilde{\Gamma}_2$  is a closed grid path in  $\mathcal{R}$ .)

Now we choose a point  $\alpha$  in  $\mathcal{R}$  and define a function  $F: \mathcal{R} \rightarrow \mathbb{C}$  by

$$F(z) = \int_{\alpha}^z f(w) dw \quad (z \in \mathcal{R}).$$

We claim that

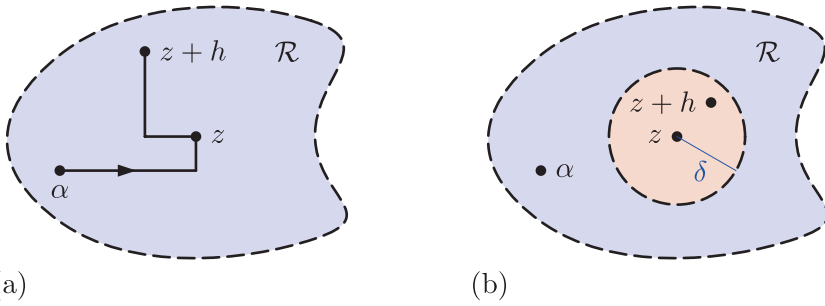
$$F'(z) = f(z), \quad \text{for } z \in \mathcal{R}, \quad (5.4)$$

so  $F$  is a primitive of  $f$  on  $\mathcal{R}$ .

To prove equation (5.4), note that if  $z + h \in \mathcal{R}$ , then

$$\begin{aligned} \frac{F(z+h) - F(z)}{h} &= \frac{1}{h} \left( \int_{\alpha}^{z+h} f(w) dw - \int_{\alpha}^z f(w) dw \right) \\ &= \frac{1}{h} \int_z^{z+h} f(w) dw. \end{aligned}$$

This last equality follows by considering a grid path in  $\mathcal{R}$  from  $\alpha$  to  $z + h$  that passes through  $z$ , as shown in Figure 5.8(a).



**Figure 5.8** (a) A grid path from  $\alpha$  to  $z + h$  through  $z$  (b) An open disc centred at  $z$  of radius  $\delta$

Hence

$$\frac{F(z+h) - F(z)}{h} - f(z) = \frac{1}{h} \int_z^{z+h} (f(w) - f(z)) dw,$$

since

$$\int_z^{z+h} f(z) dw = f(z) \int_z^{z+h} 1 dw = f(z)h.$$

To prove that  $F'(z) = f(z)$ , we need to show that

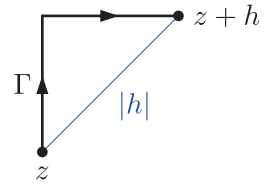
$$\lim_{h \rightarrow 0} \left( \frac{1}{h} \int_z^{z+h} (f(w) - f(z)) dw \right) = 0. \quad (5.5)$$

We do this by using the Estimation Theorem to obtain an upper estimate for the modulus of the integral.

First some preliminaries. Let  $\varepsilon$  be any positive number. Since  $f$  is continuous at  $z$ , there is a  $\delta > 0$  such that

$$|f(w) - f(z)| < \varepsilon, \quad \text{for all } z \in \mathcal{R} \text{ with } |w - z| < \delta.$$

Let us choose  $\delta$  to be sufficiently small so that the open disc centred at  $z$  of radius  $\delta$  is contained in  $\mathcal{R}$ , as illustrated in Figure 5.8(b). Consider a non-zero complex number  $h$  with  $|h| < \delta$ , and take a ‘simplest possible’ grid path  $\Gamma$  in  $\mathcal{R}$  from  $z$  to  $z + h$  (consisting of at most two line segments, see Figure 5.9).



**Figure 5.9** Simple grid path from  $z$  to  $z + h$

Now we apply the Estimation Theorem. Observe that

$$L(\Gamma) \leq 2|h|,$$

as illustrated in Figure 5.9. Next, if  $w \in \Gamma$ , then  $|w - z| < \delta$ , so  $|f(w) - f(z)| < \varepsilon$ . Applying the Estimation Theorem with  $M = \varepsilon$  and  $L \leq 2|h|$  gives

$$\left| \frac{1}{h} \int_z^{z+h} (f(w) - f(z)) dw \right| \leq \frac{1}{|h|} \times \varepsilon \times 2|h| = 2\varepsilon.$$

In summary, we have shown that for any  $\varepsilon > 0$ , there is a  $\delta > 0$  such that

$$|h| < \delta \implies \left| \frac{1}{h} \int_z^{z+h} (f(w) - f(z)) dw \right| < 2\varepsilon.$$

This establishes equation (5.5), so  $F'(z) = f(z)$ , as required. ■

At long last we can now prove Cauchy's Theorem!

### Theorem 1.2 Cauchy's Theorem

Let  $\mathcal{R}$  be a simply connected region, and let  $f$  be a function that is analytic on  $\mathcal{R}$ . Then

$$\int_{\Gamma} f(z) dz = 0,$$

for any closed contour  $\Gamma$  in  $\mathcal{R}$ .

**Proof** By the Primitive Theorem (Theorem 5.3),  $f$  has a primitive on  $\mathcal{R}$ . Also,  $f$  is analytic on  $\mathcal{R}$ , so it is continuous on  $\mathcal{R}$ . Since  $f$  is continuous on  $\mathcal{R}$  and has a primitive on  $\mathcal{R}$ , it follows from the Closed Contour Theorem (Theorem 3.4 of Unit B1) that

$$\int_{\Gamma} f(z) dz = 0,$$

as required. ■

## 5.4 Morera's Theorem

In this subsection we state and prove a theorem first established in 1886 by the Italian mathematician Giacinto Morera (1856–1909), which will be used later in the module. (The name Morera is commonly pronounced 'more-rare-ruh'.) As you can see, this theorem is a partial converse to Cauchy's Theorem.

**Theorem 5.4 Morera's Theorem**

Let  $f$  be a function that is continuous on a region  $\mathcal{R}$  and satisfies

$$\int_{\Gamma} f(z) dz = 0,$$

for all rectangular contours  $\Gamma$  in  $\mathcal{R}$ . Then  $f$  is analytic on  $\mathcal{R}$ .

**Proof** Lemma 5.1 tells us that  $f$  has a primitive  $F$  on  $\mathcal{R}$ . Then, by the Analyticity of Derivatives (Theorem 3.3), we see that the function  $f = F'$  must be analytic on  $\mathcal{R}$ , as required. ■

## Solutions to exercises

### Solution to Exercise 1.1

The regions in parts (a), (b), (d), (e), (g) and (h) are simply connected (because each one has no holes in it); the regions in parts (c) and (f) are not simply connected (because each has at least one hole in it: two in the case of (c) and one in the case of (f)).

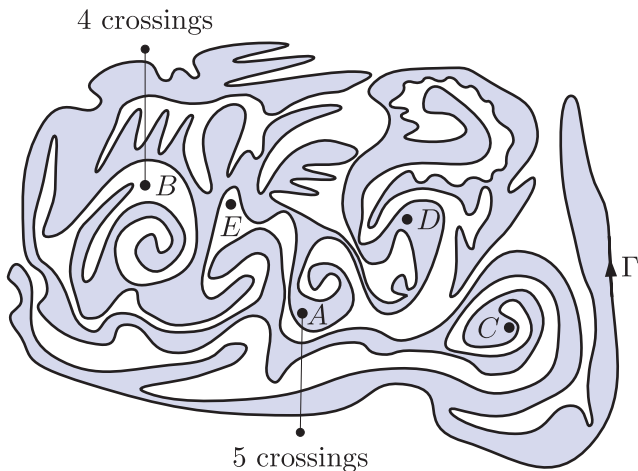
### Solution to Exercise 1.2

(a) The points  $A$ ,  $C$  and  $D$  lie inside  $\Gamma$ .

(b) There are several possible algorithms. The following algorithm makes use of the fact that, since  $\Gamma$  is simple-closed, it lies in some closed disc  $D = \{z : |z| \leq r\}$ , say, so any point in  $\mathbb{C} - D$  lies outside  $\Gamma$ .

Suppose then that we wish to determine whether a point  $\alpha$  lies inside  $\Gamma$ . To do so, choose a point  $\beta$  that definitely lies outside  $\Gamma$ . Draw a line segment from  $\alpha$  to  $\beta$ , and count the number of times that the line crosses  $\Gamma$ . If this number is *odd*, then  $\alpha$  lies inside  $\Gamma$ ; if it is *even*, then  $\alpha$  lies outside  $\Gamma$ .

The algorithm is demonstrated for the points  $A$  and  $B$  in the figure below.



### Solution to Exercise 1.3

Let  $\mathcal{R}$  be the open unit disc  $\{z : |z| < 1\}$ , a simply connected region.

(a) No,  $f$  is not analytic on  $\mathcal{R}$ .

(b) Yes, even though  $\Gamma$  is not simple-closed.

(c) No,  $f$  is not analytic on  $\mathcal{R}$ .

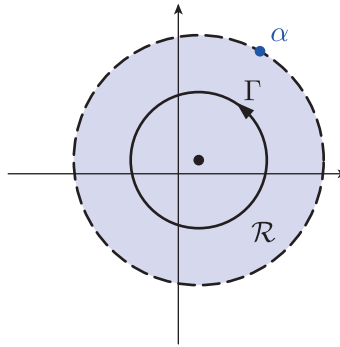
(d) Yes,  $f$  is analytic on  $\mathcal{R}$  since 3 lies outside  $\mathcal{R}$ .

(e) No,  $\Gamma$  is not closed.

(f) No,  $f$  is not analytic on  $\mathcal{R}$ .

### Solution to Exercise 1.4

Let  $\mathcal{R}$  be a simply connected region that contains  $\Gamma$ , but excludes the point  $\alpha$ . For example,  $\mathcal{R}$  could be the open disc with the same centre as  $\Gamma$  and with  $\alpha$  on its boundary.



Then  $\Gamma$  is a closed contour in  $\mathcal{R}$ , and  $f(z) = 1/(z - \alpha)$  is analytic on  $\mathcal{R}$ . Thus the conditions of Cauchy's Theorem are satisfied, so

$$\int_{\Gamma} \frac{1}{z - \alpha} dz = 0.$$

### Solution to Exercise 1.5

Let  $\Gamma = \Gamma_1 + \tilde{\Gamma}_2$ . Then  $\Gamma$  is a closed contour in  $\mathcal{R}$ , and it follows from Cauchy's Theorem that

$$\int_{\Gamma_1} f(z) dz + \int_{\tilde{\Gamma}_2} f(z) dz = \int_{\Gamma} f(z) dz = 0.$$

But, by the Reverse Contour Theorem (Theorem 2.3 of Unit B1),

$$\int_{\tilde{\Gamma}_2} f(z) dz = - \int_{\Gamma_2} f(z) dz.$$

Thus

$$\int_{\Gamma_1} f(z) dz = \int_{\Gamma_2} f(z) dz.$$

## Solution to Exercise 1.6

By the Shrinking Contour Theorem, we can replace  $\Gamma$  by any circle lying inside  $\Gamma$ , for example, by the unit circle  $C$ . Thus

$$\int_{\Gamma} \frac{1}{z} dz = \int_C \frac{1}{z} dz = 2\pi i.$$

(The value of the integral of  $1/z$  around  $C$  was mentioned at the beginning of Subsection 1.1, and was originally found in Example 2.3 of Unit B1.)

## Solution to Exercise 1.7

(a) The region  $\{z : 2 < |z - 3| < 4\}$  is an annulus, so it has a hole; therefore it is not simply connected.

(b) The region  $\{z : |z| > 0\}$  is a punctured plane, so it has a hole (at 0); therefore it is not simply connected.

(c) The region  $\{z : -1 < \operatorname{Re} z < 1\}$  is a vertical strip, so it has no holes; therefore it is simply connected.

(d) The region  $\{z : -\pi < \operatorname{Arg} z < \pi\}$  is a cut plane, so it has no holes; therefore it is simply connected.

(e) The domain of the tangent function is the region

$$\mathbb{C} - \left\{ \left(n + \frac{1}{2}\right)\pi : n \in \mathbb{Z} \right\},$$

which has holes at each of the points  $(n + \frac{1}{2})\pi$ ,  $n \in \mathbb{Z}$ ; therefore it is not simply connected.

## Solution to Exercise 1.8

(a) The path  $\Gamma : \gamma(t) = e^{it}$  ( $t \in [0, \pi]$ ) is the upper half of the circle  $\{z : |z| = 1\}$ , with endpoints  $\gamma(0) = 1$  and  $\gamma(\pi) = -1$ . Thus  $\Gamma$  is not closed and hence not simple-closed.

(b) The path  $\Gamma : \gamma(t) = e^{it}$  ( $t \in [0, 2\pi]$ ) is the circle  $\{z : |z| = 1\}$  traversed once anticlockwise. Hence  $\Gamma$  is simple-closed.

(c) The path  $\Gamma : \gamma(t) = e^{it}$  ( $t \in [0, 4\pi]$ ) is the circle  $\{z : |z| = 1\}$  traversed twice anticlockwise. Thus  $\gamma$  is not one-to-one on the interval  $[0, 4\pi)$  (for example,  $\gamma(\pi) = \gamma(3\pi) = -1$ ). Hence  $\Gamma$  is not simple-closed.

(d) The path  $\Gamma : \gamma(t) = te^{it}$  ( $t \in [0, 2\pi]$ ) is not simple-closed because it is not closed ( $\gamma(0) = 0$ , but  $\gamma(2\pi) = 2\pi$ ).

## Solution to Exercise 1.9

In each case the integrand is analytic on the simply connected region  $\mathcal{R}$  specified below, and each region  $\mathcal{R}$  contains the closed contour  $\Gamma = \{z : |z| = 2\}$ . Hence, by Cauchy's Theorem, the value of each integral is zero.

(a)  $\mathcal{R} = \mathbb{C}$

(b)  $\mathcal{R} = \{z : |z| < \pi\}$

(c)  $\mathcal{R} = \{z : |z| < \pi\}$

(d)  $\mathcal{R} = \{z : \operatorname{Im} z > -3\}$

(Other choices for  $\mathcal{R}$  are possible, of course.)

## Solution to Exercise 1.10

(a) The function  $\sec$  is not defined at  $\pi/2$ , which lies inside  $\Gamma = \{z : |z| = 2\}$ .

(b) The function  $z \mapsto \operatorname{Log}(1 + z)$  is not analytic at  $-1$ , which lies on  $\Gamma = \{z : |z| = 1\}$ .

(c) The function  $z \mapsto 1/(z - 1)$  is not defined at 1, which lies inside  $\Gamma = \{z : |z| = 3\}$ .

(d) The path  $\Gamma : \gamma(t) = (1 - t) + it$  ( $t \in [0, 1]$ ) is the line segment from 1 to  $i$ , so it is not a closed contour.

## Solution to Exercise 1.11

(a) Let  $\mathcal{R} = \mathbb{C}$ . Then  $\mathcal{R}$  is a simply connected region on which  $f(z) = ze^z$  is analytic. Also,  $\Gamma_1$  is the line segment from 0 to  $i$  and  $\Gamma_2$  is a semicircle with centre  $\frac{1}{2}i$ , initial point 0 and final point  $i$ . Thus  $\Gamma_1$  and  $\Gamma_2$  are contours in  $\mathcal{R}$  with the same initial and final points.

Hence, by the Contour Independence Theorem,

$$\int_{\Gamma_1} ze^z dz = \int_{\Gamma_2} ze^z dz.$$

(b) Let  $\mathcal{R} = \{z : \operatorname{Re} z > 0\}$ . Then  $\mathcal{R}$  is simply connected and  $g(z) = (\operatorname{Log} z)/(z - 3)$  is analytic on  $\mathcal{R} - \{3\}$ . Also,  $\Gamma_1$  is a simple-closed contour in  $\mathcal{R}$  (the circle with centre 4 and radius 2), and  $\Gamma_2$  is a simple-closed contour in  $\mathcal{R}$  (the circle with centre 3 and radius  $\frac{1}{2}$ ), and  $\Gamma_2$  lies inside  $\Gamma_1$ .

Hence, by the Shrinking Contour Theorem,

$$\int_{\Gamma_1} \frac{\operatorname{Log} z}{z-3} dz = \int_{\Gamma_2} \frac{\operatorname{Log} z}{z-3} dz.$$

### Solution to Exercise 2.1

(a) We use Cauchy's Integral Formula with  $f(z) = \sin z$ ,  $\alpha = -i$  and  $\mathcal{R} = \mathbb{C}$ . Then  $\mathcal{R}$  is simply connected,  $\Gamma = \{z : |z| = 2\}$  is a simple-closed contour in  $\mathcal{R}$ , and  $\alpha$  lies inside  $\Gamma$ . Also,  $f$  is analytic on  $\mathcal{R}$ .

It follows from Cauchy's Integral Formula that

$$\begin{aligned} \int_{\Gamma} \frac{\sin z}{z+i} dz &= 2\pi i f(-i) \\ &= 2\pi i \sin(-i) \\ &= 2\pi \sinh 1. \end{aligned}$$

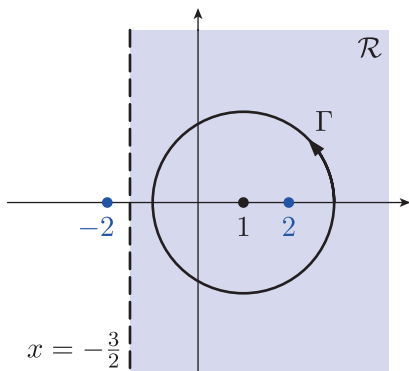
(b) We use Cauchy's Integral Formula with  $f(z) = 3z$ ,  $\alpha = -1$  and  $\mathcal{R} = \mathbb{C}$ . Then  $\mathcal{R}$  is simply connected,  $\Gamma = \{z : |z-3| = 5\}$  is a simple-closed contour in  $\mathcal{R}$ , and  $\alpha$  lies inside  $\Gamma$ . Also,  $f$  is analytic on  $\mathcal{R}$ .

It follows from Cauchy's Integral Formula that

$$\begin{aligned} \int_{\Gamma} \frac{3z}{z+1} dz &= 2\pi i f(-1) \\ &= -6\pi i. \end{aligned}$$

### Solution to Exercise 2.2

(a) The integrand is not defined at the points 2 (inside  $\Gamma$ ) and  $-2$  (outside  $\Gamma$ ). We therefore take  $f(z) = e^{3z}/(z+2)$  and  $\alpha = 2$ , and let  $\mathcal{R}$  be any simply connected region that contains  $\Gamma$  but not the point  $-2$ ; for example,  $\mathcal{R} = \{z : \operatorname{Re} z > -\frac{3}{2}\}$ . Then  $\Gamma$  is a simple-closed contour in  $\mathcal{R}$ , and  $\alpha$  lies inside  $\Gamma$ . Also,  $f$  is analytic on  $\mathcal{R}$ .

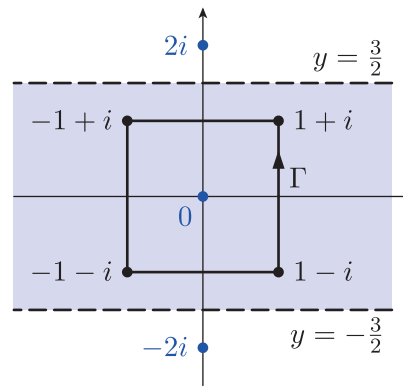


It follows from Cauchy's Integral Formula that

$$\begin{aligned} \int_{\Gamma} \frac{e^{3z}}{z^2-4} dz &= \int_{\Gamma} \frac{f(z)}{z-2} dz \\ &= 2\pi i f(2) \\ &= 2\pi i \times \left( \frac{e^6}{4} \right) = \frac{1}{2}\pi e^6 i. \end{aligned}$$

(b) The integrand is not defined at the points 0 (inside  $\Gamma$ ) and  $2i$  and  $-2i$  (both outside  $\Gamma$ ).

We therefore take  $f(z) = (\cos 2z)/(z^2+4)$  and  $\alpha = 0$ , and let  $\mathcal{R}$  be any simply connected region that contains  $\Gamma$  but not the points  $2i$  and  $-2i$ ; for example,  $\mathcal{R} = \{z : -\frac{3}{2} < \operatorname{Im} z < \frac{3}{2}\}$ . Then  $\Gamma$  is a simple-closed contour in  $\mathcal{R}$ , and  $\alpha$  lies inside  $\Gamma$ . Also,  $f$  is analytic on  $\mathcal{R}$ .



It follows from Cauchy's Integral Formula that

$$\begin{aligned} \int_{\Gamma} \frac{\cos 2z}{z(z^2+4)} dz &= \int_{\Gamma} \frac{f(z)}{z-0} dz \\ &= 2\pi i f(0) \\ &= 2\pi i \times \frac{\cos 0}{(0^2+4)} = \frac{\pi i}{2}. \end{aligned}$$

### Solution to Exercise 2.3

(a) We have

$$\frac{1}{z^2-z} = \frac{1}{z(z-1)} = \frac{A}{z} + \frac{B}{z-1},$$

where  $A$  and  $B$  are constants. Multiplying both sides by  $z^2-z$ , we obtain

$$1 = A(z-1) + Bz. \quad (\text{S1})$$

By equating the coefficients of  $z$  and constants, we obtain

$$\begin{aligned} z : \quad 0 &= A + B, \\ 1 : \quad 1 &= -A. \end{aligned}$$

Solving these simultaneous equations gives  $A = -1$  and  $B = 1$ . Thus

$$\frac{1}{z^2 - z} = -\frac{1}{z} + \frac{1}{z - 1}.$$

(Alternatively, substituting  $z = 0$  in equation (S1) gives  $A = -1$ , and substituting  $z = 1$  gives  $B = 1$ .)

(b) We have

$$\frac{1}{z(z-2)} = \frac{A}{z} + \frac{B}{z-2},$$

where  $A$  and  $B$  are constants. Multiplying both sides by  $z(z-2)$ , we obtain

$$1 = A(z-2) + Bz.$$

By equating the coefficients of  $z$  and constants, we obtain

$$z: \quad 0 = A + B,$$

$$1: \quad 1 = -2A.$$

Solving these simultaneous equations gives  $A = -\frac{1}{2}$  and  $B = \frac{1}{2}$ . Thus

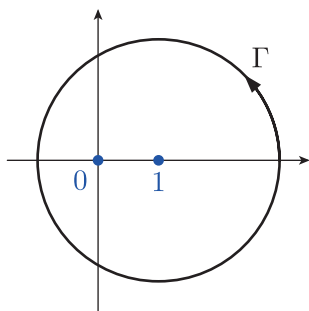
$$\frac{1}{z(z-2)} = -\frac{1}{2z} + \frac{1}{2(z-2)}.$$

## Solution to Exercise 2.4

First note that

$$z^2 - z = z(z-1),$$

and both 0 and 1 lie inside  $\Gamma$ .



Using partial fractions, we can write

$$\frac{1}{z^2 - z} = -\frac{1}{z} + \frac{1}{z - 1},$$

by Exercise 2.3(a), so

$$\int_{\Gamma} \frac{\cos 3z}{z^2 - z} dz = - \int_{\Gamma} \frac{\cos 3z}{z} dz + \int_{\Gamma} \frac{\cos 3z}{z - 1} dz.$$

We now use Cauchy's Integral Formula with  $f(z) = \cos 3z$ ,  $\mathcal{R} = \mathbb{C}$  and  $\alpha = 0$  and 1 (in turn). Then  $\mathcal{R}$  is simply connected,  $\Gamma$  is a simple-closed contour in  $\mathcal{R}$ , and  $f$  is analytic on  $\mathcal{R}$ . We obtain

$$\int_{\Gamma} \frac{\cos 3z}{z} dz = 2\pi i f(0) = 2\pi i$$

and

$$\int_{\Gamma} \frac{\cos 3z}{z - 1} dz = 2\pi i f(1) = 2\pi i \cos 3.$$

Thus

$$\begin{aligned} \int_{\Gamma} \frac{\cos 3z}{z^2 - z} dz &= -2\pi i + 2\pi i \cos 3 \\ &= 2\pi i (\cos 3 - 1). \end{aligned}$$

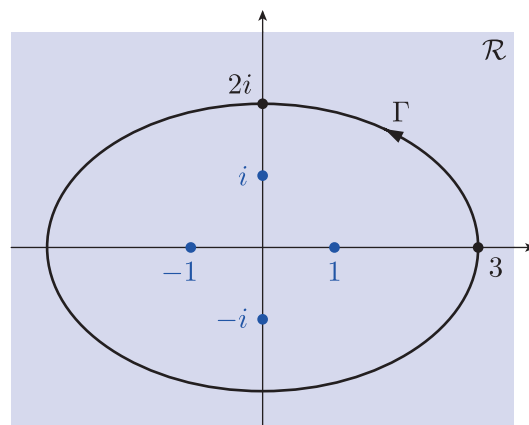
## Solution to Exercise 2.5

(a) We have

$$\begin{aligned} \int_{\Gamma} \frac{e^z}{z^4 - 1} dz &= \frac{1}{4} \int_{\Gamma} \frac{e^z}{z - 1} dz - \frac{1}{4} \int_{\Gamma} \frac{e^z}{z + 1} dz \\ &\quad + \frac{i}{4} \int_{\Gamma} \frac{e^z}{z - i} dz - \frac{i}{4} \int_{\Gamma} \frac{e^z}{z + i} dz. \end{aligned}$$

(i) All four points 1,  $-1$ ,  $i$  and  $-i$  lie inside  $\Gamma$ .

We use Cauchy's Integral Formula with  $f(z) = e^z$ ,  $\mathcal{R} = \mathbb{C}$  and  $\alpha = 1, -1, i$  and  $-i$  (in turn).



Then  $\mathcal{R}$  is simply connected,  $\Gamma$  is a simple-closed contour in  $\mathcal{R}$ , and  $f$  is analytic on  $\mathcal{R}$ . Thus

$$\begin{aligned} \int_{\Gamma} \frac{e^z}{z - 1} dz &= 2\pi i f(1) = 2\pi i e, \\ \int_{\Gamma} \frac{e^z}{z + 1} dz &= 2\pi i f(-1) = 2\pi i e^{-1}, \\ \int_{\Gamma} \frac{e^z}{z - i} dz &= 2\pi i f(i) = 2\pi i e^i, \\ \int_{\Gamma} \frac{e^z}{z + i} dz &= 2\pi i f(-i) = 2\pi i e^{-i}. \end{aligned}$$

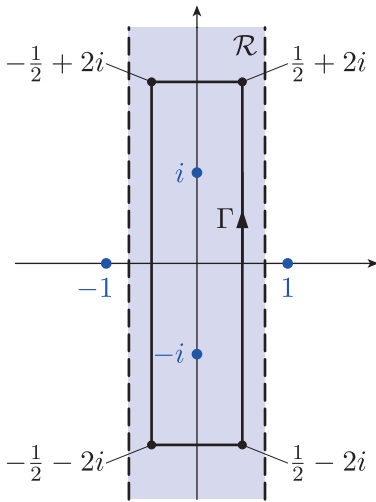
Putting all this together, we obtain

$$\begin{aligned}\int_{\Gamma} \frac{e^z}{z^4 - 1} dz &= \frac{1}{4}(2\pi i (e - e^{-1} + ie^i - ie^{-i})) \\ &= \pi i (\sinh 1 - \sin 1).\end{aligned}$$

(ii) In this part, only the two points  $i$  and  $-i$  lie inside  $\Gamma$ . We now let  $\mathcal{R}$  be any simply connected region containing  $\Gamma$  but not the points  $1$  and  $-1$ ; for example  $\mathcal{R} = \{z : -\frac{3}{4} < \operatorname{Re} z < \frac{3}{4}\}$ . The value of the integral is therefore

$$\frac{1}{4}(2\pi i (ie^i - ie^{-i})) = -\pi i \sin 1.$$

(The other two terms each have value  $0$ , by Cauchy's Theorem.)



(b) First note that

$$\begin{aligned}z^4 - 1 &= (z^2 - 1)(z^2 + 1) \\ &= (z - 1)(z + 1)(z - i)(z + i)\end{aligned}$$

and that  $i$  and  $-i$  lie inside  $\Gamma$ , but  $1$  and  $-1$  lie outside  $\Gamma$ . Also,

$$\frac{1}{z^2 + 1} = \frac{-i/2}{z - i} + \frac{i/2}{z + i},$$

by equation (2.3), so

$$\begin{aligned}\int_{\Gamma} \frac{e^z}{z^4 - 1} dz &= -\frac{i}{2} \int_{\Gamma} \frac{e^z/(z^2 - 1)}{z - i} dz \\ &\quad + \frac{i}{2} \int_{\Gamma} \frac{e^z/(z^2 - 1)}{z + i} dz.\end{aligned}$$

We now let  $\mathcal{R}$  be any simply connected region containing  $\Gamma$ , but not the points  $1$  and  $-1$ ; for example,  $\mathcal{R} = \{z : -\frac{3}{4} < \operatorname{Re} z < \frac{3}{4}\}$ .

Applying Cauchy's Integral Formula with  $f(z) = e^z/(z^2 - 1)$ , which is analytic on  $\mathcal{R}$ ,

and  $\alpha = i$  and  $-i$  (in turn) gives

$$\begin{aligned}\int_{\Gamma} \frac{e^z}{z^4 - 1} dz &= -\frac{i}{2} \times 2\pi i f(i) + \frac{i}{2} \times 2\pi i f(-i) \\ &= \pi \left( \frac{e^i}{-2} - \frac{e^{-i}}{-2} \right) \\ &= -\pi i \sin 1,\end{aligned}$$

as before.

## Solution to Exercise 2.6

We apply Cauchy's Integral Formula with  $\Gamma = C$  and  $\alpha$  the centre of  $C$ . Then

$$f(\alpha) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - \alpha} dz.$$

The standard parametrisation of  $C$  is

$$\gamma(t) = \alpha + re^{it} \quad (t \in [0, 2\pi]),$$

so

$$\begin{aligned}f(\alpha) &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(\alpha + re^{it})}{re^{it}} \times rie^{it} dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(\alpha + re^{it}) dt.\end{aligned}$$

## Solution to Exercise 2.7

It is not true that  $|\sin z| \leq 1$  for all  $z \in \mathbb{C}$ ; for example,

$$|\sin i| = |i \sinh 1| = \sinh 1 = 1.175$$

to three decimal places.

## Solution to Exercise 2.8

We use the method of proof by contradiction. Assume that the function

$$f(z) = \exp(i|z|)$$

is an entire function. Observe that

$$\exp(i|z|) = \cos |z| + i \sin |z|.$$

Hence  $|f(z)| = 1$ , so  $f$  is bounded. Therefore, by Liouville's Theorem,  $f$  is a constant function. But, for example,

$$f(0) = \exp 0 = 1$$

and

$$f(1) = \exp i = \cos 1 + i \sin 1.$$

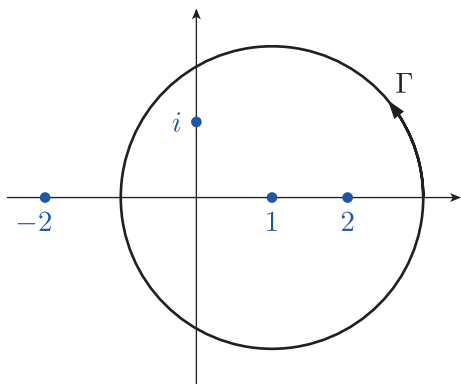
Thus  $f(0) \neq f(1)$ , so we have a contradiction. It follows that  $f$  is not an entire function.

### Solution to Exercise 2.9

Let  $p$  be a non-constant polynomial function, and let  $w \in \mathbb{C}$ . Then  $f(z) = p(z) - w$  is a non-constant polynomial function, so, by the Fundamental Theorem of Algebra, there is a complex number  $\alpha$  for which  $f(\alpha) = 0$ . Hence  $p(\alpha) = w$ , so  $w$  lies in the image set of  $p$ . Since the point  $w$  was chosen arbitrarily in  $\mathbb{C}$ , we see that the image set of  $p$  is the entire complex plane.

### Solution to Exercise 2.10

Cauchy's Integral Formula is appropriate for all parts except part (c), in which we apply Cauchy's Theorem. The figure shows  $\Gamma = \{z : |z - 1| = 2\}$  and points relevant to parts (a), (b), (d) and (e).



(a) Let  $\mathcal{R} = \mathbb{C}$ , which is simply connected. Then  $f(z) = e^{i\pi z/2}$  is analytic on  $\mathcal{R}$ ,  $\Gamma = \{z : |z - 1| = 2\}$  is a simple-closed contour in  $\mathcal{R}$ , and the point 1 lies inside  $\Gamma$ . Hence, by Cauchy's Integral Formula,

$$\begin{aligned} \int_{\Gamma} \frac{e^{i\pi z/2}}{z-1} dz &= 2\pi i f(1) \\ &= 2\pi i e^{i\pi/2} = -2\pi. \end{aligned}$$

(b) Let  $\mathcal{R} = \mathbb{C}$ , which is simply connected. Then  $f(z) = z^3$  is analytic on  $\mathcal{R}$ ,  $\Gamma = \{z : |z - 1| = 2\}$  is a simple-closed contour in  $\mathcal{R}$ , and the point 2 lies inside  $\Gamma$ . Hence, by Cauchy's Integral Formula,

$$\begin{aligned} \int_{\Gamma} \frac{z^3}{z-2} dz &= 2\pi i f(2) \\ &= 2\pi i \times 2^3 = 16\pi i. \end{aligned}$$

(c) Let  $\mathcal{R} = \{z : \operatorname{Re} z < 4\}$ , which is simply connected. Then  $f(z) = (z+4)/(z-4)$  is analytic

on  $\mathcal{R}$ , and  $\Gamma = \{z : |z - 1| = 2\}$  is a closed contour in  $\mathcal{R}$ . Hence, by Cauchy's Theorem,

$$\int_{\Gamma} \frac{z+4}{z-4} dz = 0.$$

(d) Let  $\mathcal{R} = \mathbb{C}$ , which is simply connected. Then  $f(z) = \sin z$  is analytic on  $\mathcal{R}$ ,  $\Gamma = \{z : |z - 1| = 2\}$  is a simple-closed contour in  $\mathcal{R}$ , and the point  $i$  lies inside  $\Gamma$ . Hence, by Cauchy's Integral Formula,

$$\begin{aligned} \int_{\Gamma} \frac{\sin z}{z-i} dz &= 2\pi i f(i) \\ &= 2\pi i \sin i = -2\pi \sinh 1. \end{aligned}$$

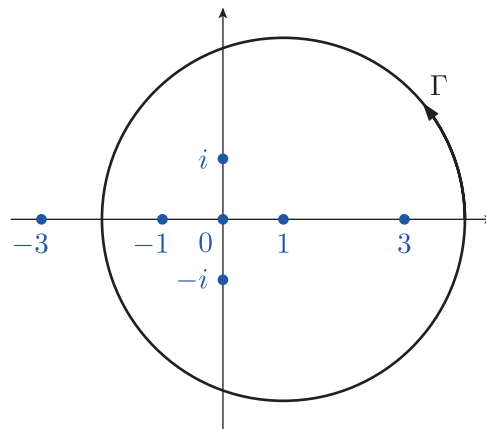
(e) First note that  $z^2 - 4 = (z+2)(z-2)$  and that the point 2 lies inside  $\Gamma = \{z : |z - 1| = 2\}$  and the point  $-2$  lies outside  $\Gamma$ .

Let  $\mathcal{R} = \{z : \operatorname{Re} z > -2\}$ , which is simply connected. Then  $f(z) = z^2/(z+2)$  is analytic on  $\mathcal{R}$ ,  $\Gamma$  is a simple-closed contour in  $\mathcal{R}$ , and the point 2 lies inside  $\Gamma$ . Hence, by Cauchy's Integral Formula,

$$\begin{aligned} \int_{\Gamma} \frac{z^2}{z^2-4} dz &= \int_{\Gamma} \frac{z^2/(z+2)}{z-2} dz \\ &= 2\pi i f(2) \\ &= 2\pi i \times \frac{4}{4} = 2\pi i. \end{aligned}$$

### Solution to Exercise 2.11

The figure shows  $\Gamma = \{z : |z - 1| = 3\}$  and the points relevant to parts (a), (b) and (c).



(a) Note that  $z^2 - 1 = (z-1)(z+1)$  and that the points 1 and  $-1$  lie inside  $\Gamma$ . So we find the partial fractions of  $1/(z^2 - 1)$ :

$$\frac{1}{z^2 - 1} = \frac{1/2}{z-1} - \frac{1/2}{z+1}.$$

Then

$$\int_{\Gamma} \frac{2z}{z^2 - 1} dz = \int_{\Gamma} \frac{z}{z - 1} dz - \int_{\Gamma} \frac{z}{z + 1} dz. \quad (\text{S2})$$

Let  $\mathcal{R} = \mathbb{C}$ , which is simply connected. Then  $f(z) = z$  is analytic on  $\mathcal{R}$ ,  $\Gamma = \{z : |z - 1| = 3\}$  is a simple-closed contour in  $\mathcal{R}$ , and the points 1 and  $-1$  lie inside  $\Gamma$ . Hence, by Cauchy's Integral Formula applied to the two integrals on the right-hand side of equation (S2),

$$\begin{aligned} \int_{\Gamma} \frac{2z}{z^2 - 1} dz &= 2\pi i f(1) - 2\pi i f(-1) \\ &= 2\pi i \times 1 - 2\pi i \times (-1) = 4\pi i. \end{aligned}$$

(b) Note that  $z^2 + 1 = (z + i)(z - i)$  and that the points  $-i$  and  $i$  lie inside  $\Gamma$ . So we find the partial fractions of  $1/(z^2 + 1)$ :

$$\frac{1}{z^2 + 1} = \frac{i/2}{z + i} - \frac{i/2}{z - i},$$

as obtained in equation (2.3). Then

$$\int_{\Gamma} \frac{\sin 2z}{z^2 + 1} dz = \frac{i}{2} \int_{\Gamma} \frac{\sin 2z}{z + i} dz - \frac{i}{2} \int_{\Gamma} \frac{\sin 2z}{z - i} dz. \quad (\text{S3})$$

Let  $\mathcal{R} = \mathbb{C}$ , which is simply connected. Then  $f(z) = \sin 2z$  is analytic on  $\mathcal{R}$ ,  $\Gamma = \{z : |z - 1| = 3\}$  is a simple-closed contour in  $\mathcal{R}$ , and the points  $-i$  and  $i$  lie inside  $\Gamma$ . Hence, by Cauchy's Integral Formula applied to the two integrals on the right-hand side of equation (S3),

$$\begin{aligned} \int_{\Gamma} \frac{\sin 2z}{z^2 + i} dz &= \frac{i}{2} \times 2\pi i f(-i) - \frac{i}{2} \times 2\pi i f(i) \\ &= -\pi \sin(-2i) + \pi \sin 2i \\ &= 2\pi i \sinh 2. \end{aligned}$$

(c) Note that  $z(z^2 - 9) = z(z - 3)(z + 3)$ , and that the points 0 and 3 lie inside  $\Gamma$ , but the point  $-3$  lies outside  $\Gamma$ . So we find the partial fractions of  $1/(z(z - 3))$ :

$$\frac{1}{z(z - 3)} = \frac{1/3}{z - 3} - \frac{1/3}{z}.$$

Then

$$\begin{aligned} \int_{\Gamma} \frac{6 \cosh z}{z(z^2 - 9)} dz &= 2 \int_{\Gamma} \frac{(\cosh z)/(z + 3)}{z - 3} dz \\ &\quad - 2 \int_{\Gamma} \frac{(\cosh z)/(z + 3)}{z} dz. \quad (\text{S4}) \end{aligned}$$

Let  $\mathcal{R} = \{z : \operatorname{Re} z > -\frac{5}{2}\}$ , which is simply connected. Then  $f(z) = (\cosh z)/(z + 3)$  is analytic on  $\mathcal{R}$ ,  $\Gamma = \{z : |z - 1| = 3\}$  is a simple-closed

contour in  $\mathcal{R}$ , and the points 0 and 3 lie inside  $\Gamma$ . Hence, by Cauchy's Integral Formula applied to the two integrals on the right-hand side of equation (S4),

$$\begin{aligned} \int_{\Gamma} \frac{6 \cosh z}{z(z^2 - 9)} dz &= 2 \times 2\pi i f(3) - 2 \times 2\pi i f(0) \\ &= 4\pi i \frac{\cosh 3}{6} - 4\pi i \frac{\cosh 0}{3} \\ &= \frac{2\pi}{3} (\cosh 3 - 2)i. \end{aligned}$$

## Solution to Exercise 2.12

(a) Let  $g = 1/f$ . Then  $g$  is entire, since  $f$  is entire and never takes the value 0. Also,  $g$  is bounded, since

$$|g(z)| = 1/|f(z)| \leq 1/K, \quad \text{for all } z \in \mathbb{C}.$$

Hence, by Liouville's Theorem,  $g$  is a constant function and therefore  $f$  is a constant function.

(b) The function  $f(z) = e^z$  is a counterexample. It is an entire function such that

$$|e^z| > 0, \quad \text{for all } z \in \mathbb{C}.$$

But  $f$  is not a constant function.

## Solution to Exercise 3.1

(a) We use Cauchy's First Derivative Formula with  $f(z) = e^{2z}$ ,  $\alpha = -1$  and  $\mathcal{R} = \mathbb{C}$ . Then  $\mathcal{R}$  is simply connected,  $\Gamma = \{z : |z| = 3\}$  is a simple-closed contour in  $\mathcal{R}$ , and  $\alpha$  lies inside  $\Gamma$ . Also,  $f$  is analytic on  $\mathcal{R}$ . It follows from Cauchy's First Derivative Formula that

$$\int_{\Gamma} \frac{e^{2z}}{(z + 1)^2} dz = 2\pi i f'(-1).$$

But  $f'(z) = 2e^{2z}$ , so  $f'(-1) = 2e^{-2}$ . Thus

$$\int_{\Gamma} \frac{e^{2z}}{(z + 1)^2} dz = 4\pi e^{-2}i.$$

(b) The integrand is not analytic at 0 and  $-1$ , both of which lie inside  $\Gamma = \{z : |z| = 3\}$ . Thus we must first express  $1/(z(z + 1)^2)$  in partial fractions. We write

$$\frac{1}{z(z + 1)^2} = \frac{A}{z} + \frac{B}{z + 1} + \frac{C}{(z + 1)^2},$$

where  $A$ ,  $B$  and  $C$  are constants. Multiplying both sides by  $z(z+1)^2$ , we obtain

$$1 = A(z+1)^2 + Bz(z+1) + Cz.$$

By equating the coefficients of  $z^2$ ,  $z$  and constants, we obtain

$$z^2 : 0 = A + B,$$

$$z : 0 = 2A + B + C,$$

$$1 : 1 = A.$$

So  $A = 1$ ,  $B = -1$ ,  $C = -1$ . Thus

$$\frac{1}{z(z+1)^2} = \frac{1}{z} - \frac{1}{z+1} - \frac{1}{(z+1)^2},$$

and hence

$$\begin{aligned} \int_{\Gamma} \frac{e^{2z}}{z(z+1)^2} dz \\ = \int_{\Gamma} \frac{e^{2z}}{z} dz - \int_{\Gamma} \frac{e^{2z}}{z+1} dz - \int_{\Gamma} \frac{e^{2z}}{(z+1)^2} dz. \end{aligned}$$

We now use Cauchy's Integral Formula and First Derivative Formula with  $f(z) = e^{2z}$ ,  $\alpha = 0$  and  $-1$ , and  $\mathcal{R} = \mathbb{C}$ . Then  $\mathcal{R}$  is simply connected,  $\Gamma$  is a simple-closed contour in  $\mathcal{R}$ , and both values of  $\alpha$  lie inside  $\Gamma$ . Also,  $f$  is analytic on  $\mathcal{R}$ .

It follows from Cauchy's Integral Formula that

$$\int_{\Gamma} \frac{e^{2z}}{z} dz = 2\pi i f(0) = 2\pi i e^0 = 2\pi i,$$

$$\int_{\Gamma} \frac{e^{2z}}{z+1} dz = 2\pi i f(-1) = 2\pi e^{-2}i,$$

and it follows from Cauchy's First Derivative Formula that

$$\begin{aligned} \int_{\Gamma} \frac{e^{2z}}{(z+1)^2} dz &= 2\pi i f'(-1) \\ &= 2\pi i \times 2e^{-2} = 4\pi e^{-2}i. \end{aligned}$$

Putting all this together, we obtain

$$\begin{aligned} \int_{\Gamma} \frac{e^{2z}}{z(z+1)^2} dz &= 2\pi i (1 - e^{-2} - 2e^{-2}) \\ &= 2\pi(1 - 3e^{-2})i. \end{aligned}$$

## Solution to Exercise 3.2

(a) We use Cauchy's  $n$ th Derivative Formula with  $n = 2$ ,  $f(z) = \cosh 2z$ ,  $\alpha = -i$  and  $\mathcal{R} = \mathbb{C}$ .

Then  $\mathcal{R}$  is simply connected,  $\Gamma = \{z : |z| = 2\}$  is a simple-closed contour in  $\mathcal{R}$ , and  $\alpha$  lies inside  $\Gamma$ .

Also,  $f$  is analytic on  $\mathcal{R}$ .

It follows from Cauchy's  $n$ th Derivative Formula that

$$\int_{\Gamma} \frac{\cosh 2z}{(z+i)^3} dz = 2\pi i \frac{f''(-i)}{2!}.$$

But  $f''(z) = 4 \cosh 2z$ , so

$$f''(-i) = 4 \cosh(-2i) = 4 \cos 2.$$

Thus

$$\int_{\Gamma} \frac{\cosh 2z}{(z+i)^3} dz = 4\pi i \cos 2.$$

(b) We use Cauchy's  $n$ th Derivative Formula with  $n = 9$ ,  $f(z) = ze^z$ ,  $\alpha = 1$  and  $\mathcal{R} = \mathbb{C}$ . Then  $\mathcal{R}$  is simply connected,  $\Gamma = \{z : |z| = 2\}$  is a simple-closed contour in  $\mathcal{R}$ , and  $\alpha$  lies inside  $\Gamma$ . Also,  $f$  is analytic on  $\mathcal{R}$ .

It follows from Cauchy's  $n$ th Derivative Formula that

$$\int_{\Gamma} \frac{ze^z}{(z-1)^{10}} dz = 2\pi i \frac{f^{(9)}(1)}{9!}.$$

But

$$f'(z) = e^z + ze^z,$$

$$f''(z) = 2e^z + ze^z,$$

$$\vdots$$

$$f^{(9)}(z) = 9e^z + ze^z,$$

so  $f^{(9)}(1) = 10e$ . Thus

$$\int_{\Gamma} \frac{ze^z}{(z-1)^{10}} dz = \frac{20e\pi i}{9!} = \frac{e\pi i}{18144}.$$

(c) The integrand is not analytic at 0 and  $-1$ , both of which lie inside  $\Gamma$ . Thus we must first express  $1/(z^3(z+1))$  in partial fractions. We write

$$\frac{1}{z^3(z+1)} = \frac{A}{z} + \frac{B}{z^2} + \frac{C}{z^3} + \frac{D}{z+1},$$

where  $A$ ,  $B$ ,  $C$  and  $D$  are constants. Multiplying both sides by  $z^3(z+1)$ , we obtain

$$1 = Az^2(z+1) + Bz(z+1) + C(z+1) + Dz^3.$$

By equating the coefficients of  $z^3$ ,  $z^2$ ,  $z$  and constants, we obtain

$$z^3 : 0 = A + D,$$

$$z^2 : 0 = A + B,$$

$$z : 0 = B + C,$$

$$1 : 1 = C.$$

So  $A = C = 1$ ,  $B = D = -1$ , and

$$\frac{1}{z^3(z+1)} = \frac{1}{z} - \frac{1}{z^2} + \frac{1}{z^3} - \frac{1}{z+1}.$$

Thus

$$\begin{aligned} \int_{\Gamma} \frac{e^{2z}}{z^3(z+1)} dz \\ = \int_{\Gamma} \frac{e^{2z}}{z} dz - \int_{\Gamma} \frac{e^{2z}}{z^2} dz + \int_{\Gamma} \frac{e^{2z}}{z^3} dz - \int_{\Gamma} \frac{e^{2z}}{z+1} dz. \end{aligned}$$

We now use Cauchy's Integral Formula and First and Second Derivative Formulas with  $f(z) = e^{2z}$ ,  $\alpha = 0$  and  $-1$ , and  $\mathcal{R} = \mathbb{C}$ . Then  $\mathcal{R}$  is simply connected,  $\Gamma = \{z : |z| = 2\}$  is a simple-closed contour in  $\mathcal{R}$ , and both values of  $\alpha$  lie inside  $\Gamma$ . Also,  $f$  is analytic on  $\mathcal{R}$ .

It follows from Cauchy's Integral Formula that

$$\begin{aligned} \int_{\Gamma} \frac{e^{2z}}{z} dz &= 2\pi i f(0) = 2\pi i, \\ \int_{\Gamma} \frac{e^{2z}}{z+1} dz &= 2\pi i f(-1) = 2\pi e^{-2}i, \end{aligned}$$

and it follows from Cauchy's First Derivative Formula that

$$\begin{aligned} \int_{\Gamma} \frac{e^{2z}}{z^2} dz &= 2\pi i f'(0) \\ &= 2\pi i \times 2e^0 = 4\pi i. \end{aligned}$$

Also, it follows from Cauchy's Second Derivative Formula that

$$\begin{aligned} \int_{\Gamma} \frac{e^{2z}}{z^3} dz &= 2\pi i \frac{f''(0)}{2!} \\ &= 2\pi i \times \frac{4e^0}{2!} = 4\pi i. \end{aligned}$$

Thus

$$\begin{aligned} \int_{\Gamma} \frac{e^{2z}}{z^3(z+1)} dz &= 2\pi i - 4\pi i + 4\pi i - 2\pi e^{-2}i \\ &= 2\pi(1 - e^{-2})i. \end{aligned}$$

### Solution to Exercise 3.3

By Cauchy's  $n$ th Derivative Formula,

$$f^{(n)}(\alpha) = \frac{n!}{2\pi i} \int_{\Gamma} \frac{f(z)}{(z-\alpha)^{n+1}} dz.$$

On  $\Gamma$  we have  $|z - \alpha| = r$  and  $|f(z)| \leq K$ . So, by applying the Estimation Theorem to this integral with  $M = K/r^{n+1}$  and  $L = 2\pi r$ , we see that

$$|f^{(n)}(\alpha)| \leq \frac{n!}{2\pi} \times \frac{K}{r^{n+1}} \times 2\pi r = \frac{Kn!}{r^n}.$$

### Solution to Exercise 3.4

(a) Assume that there exists an analytic function  $F$  such that  $F'(z) = |z|$ , for  $z \in \mathbb{C}$ . Then, by the Analyticity of Derivatives,  $F'$  is analytic on  $\mathbb{C}$ . But  $F'(z) = |z|$ , which is not analytic on  $\mathbb{C}$  (by Example 1.5 of Unit A4). This contradiction implies that no such function  $F$  exists.

(b) Since  $f$  is entire, so is  $f'$ , by the Analyticity of Derivatives. Since  $f'$  is entire and bounded,  $f'$  is constant, by Liouville's Theorem.

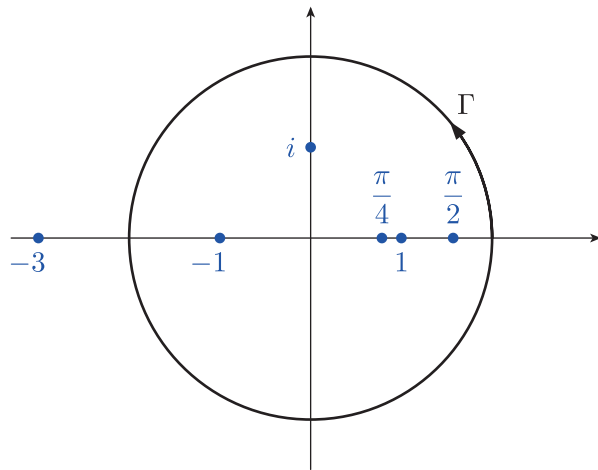
Thus  $f'(z) = \alpha$ , where  $\alpha \in \mathbb{C}$ . Then the function  $g(z) = \alpha z$  is a primitive of  $f'$  on  $\mathbb{C}$ . Furthermore, any primitive of  $f'$  on  $\mathbb{C}$  is equal to  $g(z) + \beta$ , for some constant  $\beta$  (by Exercise 3.6 of Unit B1).

Hence

$$f(z) = \alpha z + \beta, \quad \text{where } \alpha, \beta \in \mathbb{C}.$$

### Solution to Exercise 3.5

The figure shows  $\Gamma = \{z : |z| = 2\}$  and points relevant to parts (a)–(e).



For parts (a), (b), (d) and (e), we take  $\mathcal{R} = \mathbb{C}$ , which is simply connected.

(a) The function  $f(z) = \cos z$  is analytic on  $\mathcal{R}$ ,  $\Gamma$  is a simple-closed contour in  $\mathcal{R}$ , and the point  $\pi/2$  lies inside  $\Gamma$ .

Hence, by Cauchy's First Derivative Formula,

$$\begin{aligned} \int_{\Gamma} \frac{\cos z}{(z - \pi/2)^2} dz &= 2\pi i f'(\pi/2) \\ &= 2\pi i (-\sin(\pi/2)) = -2\pi i. \end{aligned}$$

(b) The function  $f(z) = \cosh \pi z$  is analytic on  $\mathcal{R}$ ,  $\Gamma$  is a simple-closed contour in  $\mathcal{R}$ , and the point  $i$  lies inside  $\Gamma$ . Hence, by Cauchy's Second Derivative Formula,

$$\begin{aligned}\int_{\Gamma} \frac{\cosh \pi z}{(z-i)^3} dz &= \frac{2\pi i}{2!} f''(i) \\ &= \pi i \times \pi^2 \cosh \pi i \\ &= \pi^3 i \cos \pi = -\pi^3 i.\end{aligned}$$

(c) Note that  $z^2 + 2z - 3 = (z+3)(z-1)$  and that the point 1 lies inside  $\Gamma$  but the point  $-3$  does not. Let  $\mathcal{R} = \{z : \operatorname{Re} z > -\frac{5}{2}\}$ . Then  $f(z) = (\sin z)/(z+3)^2$  is analytic on  $\mathcal{R}$ , and  $\Gamma$  is a simple-closed contour in  $\mathcal{R}$ . Observe that

$$f'(z) = \frac{(z+3)^2 \cos z - 2(z+3) \sin z}{(z+3)^4}.$$

Hence, by Cauchy's First Derivative Formula,

$$\begin{aligned}\int_{\Gamma} \frac{\sin z}{(z^2 + 2z - 3)^2} dz &= \int_{\Gamma} \frac{(\sin z)/(z+3)^2}{(z-1)^2} dz \\ &= 2\pi i f'(1) \\ &= 2\pi i \times \frac{4^2 \cos 1 - 8 \sin 1}{4^4} \\ &= \frac{1}{16} (2 \cos 1 - \sin 1) \pi i.\end{aligned}$$

(d) The function  $f(z) = \sin 2z$  is analytic on  $\mathcal{R}$ ,  $\Gamma$  is a simple-closed contour in  $\mathcal{R}$ , and the point  $\pi/4$  lies inside  $\Gamma$ . Hence, by Cauchy's  $n$ th Derivative Formula with  $n = 4$ ,

$$\begin{aligned}\int_{\Gamma} \frac{\sin 2z}{(z - \pi/4)^5} dz &= \frac{2\pi i}{4!} f^{(4)}(\pi/4) \\ &= \frac{\pi i}{12} \times 2^4 \sin 2\pi/4 \\ &= \frac{4\pi}{3} i.\end{aligned}$$

(e) The function  $f(z) = 1$  is analytic on  $\mathcal{R}$ ,  $\Gamma$  is a simple-closed contour in  $\mathcal{R}$ , and the point  $-1$  lies inside  $\Gamma$ . Hence, by Cauchy's  $n$ th Derivative Formula with  $n = 10$ ,

$$\int_{\Gamma} \frac{1}{(z+1)^{11}} dz = \frac{2\pi i}{10!} f^{(10)}(-1) = 0,$$

since  $f^{(10)}(z) = 0$ , for all  $z \in \mathbb{C}$ .

*Remark:* This integral can also be evaluated using the Closed Contour Theorem.

## Solution to Exercise 3.6

Note that

$$z^4 - 2z^3 + z^2 = z^2(z^2 - 2z + 1) = z^2(z-1)^2,$$

and that the points 0 and 1 lie inside  $\Gamma$ . Let us write

$$\frac{1}{z^2(z-1)^2} = \frac{A}{z} + \frac{B}{z^2} + \frac{C}{z-1} + \frac{D}{(z-1)^2},$$

for constants  $A, B, C$  and  $D$ . Multiplying both sides by  $z^2(z-1)^2$ , we obtain

$$1 = Az(z-1)^2 + B(z-1)^2 + Cz^2(z-1) + Dz^2.$$

By equating the coefficients of  $z^3, z^2, z$  and constants, we obtain

$$\begin{aligned}z^3 : \quad & 0 = A + C, \\ z^2 : \quad & 0 = -2A + B - C + D, \\ z : \quad & 0 = A - 2B, \\ 1 : \quad & 1 = B.\end{aligned}$$

Solving these simultaneous equations gives

$$A = 2, \quad B = 1, \quad C = -2, \quad D = 1.$$

Hence

$$\frac{1}{z^4 - 2z^3 + z^2} = \frac{2}{z} + \frac{1}{z^2} - \frac{2}{z-1} + \frac{1}{(z-1)^2}.$$

Thus

$$\begin{aligned}\int_{\Gamma} \frac{e^{3z}}{z^4 - 2z^3 + z^2} dz &= 2 \int_{\Gamma} \frac{e^{3z}}{z} dz + \int_{\Gamma} \frac{e^{3z}}{z^2} dz \\ &\quad - 2 \int_{\Gamma} \frac{e^{3z}}{z-1} dz \\ &\quad + \int_{\Gamma} \frac{e^{3z}}{(z-1)^2} dz.\end{aligned}$$

We now apply Cauchy's Integral Formula to the first and third integrals on the right-hand side of this equation and apply Cauchy's First Derivative Formula to the other two. We take  $\mathcal{R} = \mathbb{C}$ , which is simply connected. Then  $f(z) = e^{3z}$  is analytic on  $\mathcal{R}$ ,  $\Gamma$  is a simple-closed contour in  $\mathcal{R}$ , and the points 0 and 1 lie inside  $\Gamma$ .

We obtain

$$\begin{aligned}\int_{\Gamma} \frac{e^{3z}}{z^4 - 2z^3 + z^2} dz &= 2\pi i (2f(0) + f'(0) - 2f(1) + f'(1)) \\ &= 2\pi i (2e^0 + 3e^0 - 2e^3 + 3e^3) \\ &= 2\pi (5 + e^3) i.\end{aligned}$$

## Solution to Exercise 4.1

(a) Let

$$\frac{1}{z(z-3)} = \frac{A}{z} + \frac{B}{z-3},$$

where  $A$  and  $B$  are constants. Multiplying both sides by  $z(z-3)$ , we obtain

$$1 = A(z-3) + Bz.$$

By equating the coefficients of  $z$  and constants, we obtain

$$z : 0 = A + B,$$

$$1 : 1 = -3A.$$

Solving these simultaneous equations gives

$$A = -1/3, \quad B = 1/3.$$

Thus

$$\frac{1}{z(z-3)} = \frac{1/3}{z-3} - \frac{1/3}{z}.$$

(b) Let

$$\frac{1}{z^2(z-3)} = \frac{A}{z} + \frac{B}{z^2} + \frac{C}{z-3},$$

where  $A$ ,  $B$  and  $C$  are constants. Multiplying both sides by  $z^2(z-3)$ , we obtain

$$1 = Az(z-3) + B(z-3) + Cz^2.$$

By equating the coefficients of  $z^2$ ,  $z$  and constants, we obtain

$$z^2 : 0 = A + C,$$

$$z : 0 = -3A + B,$$

$$1 : 1 = -3B.$$

Solving these simultaneous equations gives

$$A = -1/9, \quad B = -1/3, \quad C = 1/9.$$

Thus

$$\frac{1}{z^2(z-3)} = \frac{1/9}{z-3} - \frac{1/9}{z} - \frac{1/3}{z^2}.$$

Alternatively, by part (a) we have

$$\frac{1}{z(z-3)} = \frac{1/3}{z-3} - \frac{1/3}{z}.$$

Multiplying both sides of this equation by  $1/z$  gives

$$\begin{aligned} \frac{1}{z^2(z-3)} &= \frac{1/3}{z(z-3)} - \frac{1/3}{z^2} \\ &= \frac{1}{3} \left( \frac{1/3}{z-3} - \frac{1/3}{z} \right) - \frac{1/3}{z^2} \\ &= \frac{1/9}{z-3} - \frac{1/9}{z} - \frac{1/3}{z^2}. \end{aligned}$$

(c) Let

$$\frac{1}{z^3(z-3)} = \frac{A}{z} + \frac{B}{z^2} + \frac{C}{z^3} + \frac{D}{z-3},$$

where  $A$ ,  $B$ ,  $C$  and  $D$  are constants. Multiplying both sides by  $z^3(z-3)$ , we obtain

$$1 = Az^2(z-3) + Bz(z-3) + C(z-3) + Dz^3.$$

By equating the coefficients of  $z^3$ ,  $z^2$ ,  $z$  and constants, we obtain

$$z^3 : 0 = A + D,$$

$$z^2 : 0 = -3A + B,$$

$$z : 0 = -3B + C,$$

$$1 : 1 = -3C.$$

Solving these simultaneous equations gives

$$A = -1/27, \quad B = -1/9, \quad C = -1/3, \quad D = 1/27.$$

Thus

$$\frac{1}{z^3(z-3)} = \frac{1/27}{z-3} - \frac{1/27}{z} - \frac{1/9}{z^2} - \frac{1/3}{z^3}.$$

Alternatively, by part (b) we have

$$\frac{1}{z^2(z-3)} = \frac{1/9}{z-3} - \frac{1/9}{z} - \frac{1/3}{z^2}.$$

Multiplying both sides of this equation by  $1/z$ , and using the result of part (a), we obtain

$$\begin{aligned} \frac{1}{z^3(z-3)} &= \frac{1/9}{z(z-3)} - \frac{1/9}{z^2} - \frac{1/3}{z^3} \\ &= \frac{1}{9} \left( \frac{1/3}{z-3} - \frac{1/3}{z} \right) - \frac{1/9}{z^2} - \frac{1/3}{z^3} \\ &= \frac{1/27}{z-3} - \frac{1/27}{z} - \frac{1/9}{z^2} - \frac{1/3}{z^3}. \end{aligned}$$

## Solution to Exercise 4.2

(a) Using equation (4.1), we have

$$\int_{\Gamma} \frac{1}{z} dz = 2\pi i,$$

since  $0$  lies inside  $\Gamma = \{z : |z-1| = 2\}$ .

(b) Using equation (4.1), we have

$$\int_{\Gamma} \frac{1}{z-5} dz = 0,$$

since 5 lies outside  $\Gamma = \{z : |z-1| = 2\}$ .

(c) The standard parametrisation of  $\Gamma = \{z : |z-1| = 2\}$  is

$$\gamma(t) = 1 + 2e^{it} \quad (t \in [0, 2\pi]).$$

Hence

$$\begin{aligned} \int_{\Gamma} \frac{1}{(z-1)^2} dz &= \int_0^{2\pi} \frac{1}{(2e^{it})^2} \times 2ie^{it} dt \\ &= \frac{i}{2} \int_0^{2\pi} e^{-it} dt \\ &= \frac{i}{2} \left( \int_0^{2\pi} \cos t dt - i \int_0^{2\pi} \sin t dt \right) \\ &= 0. \end{aligned}$$

This integral can also be evaluated by using the Closed Contour Theorem with  $f(z) = 1/(z-1)^2$ , which is continuous and has a primitive  $F(z) = -1/(z-1)$  on the region  $\mathcal{R} = \mathbb{C} - \{1\}$ . This region contains the simple-closed contour  $\Gamma$ .

By the Closed Contour Theorem,

$$\int_{\Gamma} \frac{1}{(z-1)^2} dz = 0.$$

### Solution to Exercise 4.3

The integrand is analytic except at 0 and 3, which lie inside  $\Gamma$ , and  $-3$ , which lies outside  $\Gamma$ . Hence we choose

$$f(z) = \frac{e^z}{z+3},$$

which is analytic on  $\mathcal{R} = \{z : \operatorname{Re} z > -2\}$ , as in Example 4.3. As noted in that example,  $\mathcal{R}$  is simply connected and  $\Gamma$  is contained in  $\mathcal{R}$ .

By Exercise 4.1(c), we have

$$\frac{1}{z^3(z-3)} = \frac{1/27}{z-3} - \frac{1/27}{z} - \frac{1/9}{z^2} - \frac{1/3}{z^3},$$

so

$$\begin{aligned} \int_{\Gamma} \frac{f(z)}{z^3(z-3)} dz &= \frac{1}{27} \int_{\Gamma} \frac{f(z)}{z-3} dz - \frac{1}{27} \int_{\Gamma} \frac{f(z)}{z} dz \\ &\quad - \frac{1}{9} \int_{\Gamma} \frac{f(z)}{z^2} dz - \frac{1}{3} \int_{\Gamma} \frac{f(z)}{z^3} dz. \end{aligned}$$

Applying Cauchy's Integral Formula to the first and second integrals on the right-hand side, Cauchy's First Derivative Formula to the third, and Cauchy's Second Derivative Formula to the fourth, gives

$$\begin{aligned} \int_{\Gamma} \frac{f(z)}{z^3(z-3)} dz &= \frac{1}{27} \times 2\pi i f(3) - \frac{1}{27} \times 2\pi i f'(0) \\ &\quad - \frac{1}{9} \times 2\pi i f''(0) - \frac{1}{3} \times \frac{2\pi i}{2!} f'''(0) \\ &= \frac{2\pi i}{27} \times \frac{e^3}{6} - \frac{2\pi i}{27} \times \frac{1}{3} \\ &\quad - \frac{2\pi i}{9} \times \frac{2}{9} - \frac{\pi i}{3} \times \frac{5}{27}, \end{aligned}$$

since

$$f'(z) = \frac{(z+2)e^z}{(z+3)^2} \quad \text{and} \quad f''(z) = \frac{(z^2+4z+5)e^z}{(z+3)^3}.$$

Simplifying this, we obtain

$$\int_{\Gamma} \frac{f(z)}{z^3(z-3)} dz = \frac{\pi}{81}(e^3 - 11)i.$$

### Solution to Exercise 4.4

(a) In this case,

$$g(z) = e^{2z} \quad \text{and} \quad p(z) = z(z^2 + 1).$$

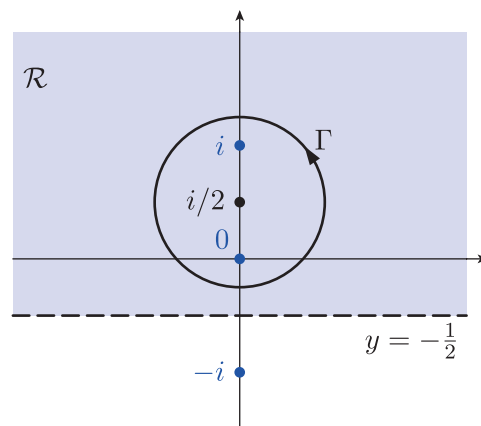
The three steps of the strategy are as follows.

1. Write  $p(z) = r(z)s(z)$ , where

$$r(z) = (z-i)z \quad \text{and} \quad s(z) = z+i.$$

The zeros  $i$  and  $0$  of  $r$  lie inside

$\Gamma = \{z : |z-i/2| = 3/4\}$ , and the zero  $-i$  of  $s$  lies outside  $\Gamma$ , as shown in the figure.



Then  $f(z) = g(z)/s(z) = e^{2z}/(z+i)$  is analytic on the simply connected region  $\mathcal{R} = \{z : \operatorname{Im} z > -1/2\}$ , which contains  $\Gamma$ .

2. Let

$$\frac{1}{r(z)} = \frac{1}{(z-i)z} = \frac{A}{z-i} + \frac{B}{z},$$

where  $A$  and  $B$  are constants. Multiplying both sides by  $(z-i)z$ , we obtain

$$1 = Az + B(z-i).$$

By equating the coefficients of  $z$  and constants, we obtain

$$z: 0 = A + B,$$

$$1: 1 = -Bi.$$

Solving these simultaneous equations gives

$A = -i$  and  $B = i$ . Thus

$$\frac{1}{r(z)} = -\frac{i}{z-i} + \frac{i}{z}.$$

3. We have

$$\begin{aligned} \int_{\Gamma} \frac{e^{2z}}{z(z^2+1)} dz \\ = -i \int_{\Gamma} \frac{e^{2z}/(z+i)}{z-i} dz + i \int_{\Gamma} \frac{e^{2z}/(z+i)}{z} dz. \end{aligned}$$

Applying Cauchy's Integral Formula to each of the integrals on the right-hand side, we obtain

$$\begin{aligned} \int_{\Gamma} \frac{e^{2z}}{z(z^2+1)} dz &= -i \times 2\pi i f(i) + i \times 2\pi i f(0) \\ &= 2\pi(e^{2i}/(2i) - 1/i) \\ &= -\pi i(e^{2i} - 2) \\ &= -\pi i(\cos 2 + i \sin 2 - 2) \\ &= \pi \sin 2 + (2 - \cos 2)\pi i. \end{aligned}$$

(b) In this case,

$$g(z) = e^{2z} \quad \text{and} \quad p(z) = z^2(z^2+1).$$

The three steps of the strategy are as follows.

1. Write  $p(z) = r(z)s(z)$ , where

$$r(z) = (z-i)z^2 \quad \text{and} \quad s(z) = z+i.$$

Again, the zeros  $i$  and  $0$  of  $r$  lie inside

$\Gamma = \{z : |z-i/2| = 3/4\}$ , and the zero  $-i$  of  $s$  lies outside  $\Gamma$ . Also,

$f(z) = g(z)/s(z) = e^{2z}/(z+i)$  is analytic on the simply connected region  $\mathcal{R} = \{z : \operatorname{Im} z > -\frac{1}{2}\}$ , which contains  $\Gamma$ .

2. As you can check,

$$\frac{1}{r(z)} = \frac{1}{(z-i)z^2} = -\frac{1}{z-i} + \frac{1}{z} + \frac{i}{z^2}.$$

3. We have

$$\begin{aligned} \int_{\Gamma} \frac{e^{2z}}{z^2(z^2+1)} dz &= - \int_{\Gamma} \frac{e^{2z}/(z+i)}{z-i} dz \\ &\quad + \int_{\Gamma} \frac{e^{2z}/(z+i)}{z} dz \\ &\quad + i \int_{\Gamma} \frac{e^{2z}/(z+i)}{z^2} dz. \end{aligned}$$

Applying Cauchy's Integral Formula to the first two integrals on the right-hand side, and applying Cauchy's First Derivative Formula to the third, we obtain

$$\begin{aligned} \int_{\Gamma} \frac{e^{2z}}{z^2(z^2+1)} dz \\ = -2\pi i f(i) + 2\pi i f'(0) + i \times 2\pi i f'(0) \\ = -2\pi i \times \frac{e^{2i}}{2i} + 2\pi i \times \frac{1}{i} - 2\pi(1-2i) \\ = -\pi \cos 2 + (4 - \sin 2)\pi i, \end{aligned}$$

where we have used the formula

$$f'(z) = (2z+2i-1)e^{2z}/(z+i)^2.$$

## Solution to Exercise 4.5

We provide only brief solutions.

(a) By Cauchy's Theorem with  $\mathcal{R} = \mathbb{C}$ ,

$$\int_{\Gamma_1} \sin z \, dz = 0.$$

(b) By Cauchy's Theorem with  $\mathcal{R} = \mathbb{C}$ ,

$$\int_{\Gamma_1} \sin(z^2) \, dz = 0.$$

(c) By Cauchy's Theorem with  $\mathcal{R} = \{z : \operatorname{Im} z > -1/2\}$ ,

$$\int_{\Gamma_1} \frac{1}{z+i} \, dz = 0.$$

(d) The integral

$$\int_{\Gamma_2} z \sin z \, dz$$

cannot be evaluated by the methods of this unit, since  $\Gamma_2$  is not a closed contour. The integral can be evaluated using Integration by Parts and the Fundamental Theorem of Calculus.

(e) By Cauchy's First Derivative Formula

with  $\mathcal{R} = \mathbb{C}$ ,

$$\begin{aligned}\int_{\Gamma_3} \frac{\sin z}{z^2} dz &= 2\pi i \sin'(0) \\ &= 2\pi i \cos 0 = 2\pi i.\end{aligned}$$

(f) By Cauchy's  $n$ th Derivative Formula

with  $n = 4$  and  $\mathcal{R} = \mathbb{C}$ ,

$$\begin{aligned}\int_{\Gamma_3} \frac{\cosh z}{z^5} dz &= \frac{2\pi i}{4!} \cosh^{(4)}(0) \\ &= \frac{\pi i}{12} \cosh 0 = \frac{\pi i}{12}.\end{aligned}$$

(g) Let  $\mathcal{R} = \{z : -\pi < \operatorname{Im} z < \pi\}$ . This region does not contain any of the zeros of  $\cosh \frac{1}{2}z$ , which have the form  $(2n+1)\pi i$ , for  $n \in \mathbb{Z}$ . Hence, by Cauchy's Theorem,

$$\int_{\Gamma_3} \frac{z^5}{\cosh \frac{1}{2}z} dz = 0.$$

(h) By Cauchy's Integral Formula with

$\mathcal{R} = \{z : \operatorname{Im} z > -1/4\}$ ,

$$\begin{aligned}\int_{\Gamma_1} \frac{z}{4z^2 + 1} dz &= \int_{\Gamma_1} \frac{z/(2z+i)}{2z-i} dz \\ &= \frac{1}{2} \int_{\Gamma_1} \frac{z/(2z+i)}{z - \frac{1}{2}i} dz \\ &= \frac{1}{2} \times 2\pi i \times \left(\frac{1}{2}i / (2 \times \frac{1}{2}i + i)\right) \\ &= \frac{\pi i}{4}.\end{aligned}$$

(i) The integral

$$\int_{\Gamma_3} \operatorname{Re} z \, dz$$

cannot be evaluated by the methods of this unit, since  $f(z) = \operatorname{Re} z$  is not an analytic function. The integral can be evaluated using parametrisation.



Unit B3

Taylor series



# Introduction

In this unit we develop an important technique for investigating the properties of analytic functions. Rather than working with a given analytic function directly, we represent the function by an expression known as a *power series* and investigate the series instead.

A power series can be thought of as an infinite-degree polynomial, such as

$$1 + 2z + 3z^2 + 4z^3 + 5z^4 + \cdots.$$

In order to make this notion of a power series precise, we need to be able to work with sums that involve infinitely many terms. In Section 1 we show how this is done by using the idea of the limit of an infinite sequence to give a rigorous definition of an infinite sum, or a *convergent series* as it is more formally known. We then discuss a number of tests that can be used to check whether a series converges.

In Section 2 we give a formal definition of a power series, and observe that power series can be used to define functions. For example, we can define a function  $f$  by the rule

$$f(z) = 1 + 2z + 3z^2 + 4z^3 + 5z^4 + \cdots.$$

We then show that we can find the derivative of such a function by differentiating the power series term by term as if it were a polynomial.

In Section 3 we turn the process around. Instead of starting with a complex power series and using it to define a function, we start with a function  $f$  that is analytic at a point  $\alpha$  and obtain a power series that represents  $f$  near  $\alpha$ . This is the *Taylor series* about  $\alpha$  for  $f$ , a version of which may be familiar to you from real analysis. The existence and uniqueness of the Taylor series about  $\alpha$  for  $f$  is established in Taylor's Theorem. After proving this theorem, we give a proof of Cauchy's  $n$ th Derivative Formula, as promised in Unit B2.

Section 4 illustrates how you can find new Taylor series by manipulating other known Taylor series. This approach is often easier than a direct application of Taylor's Theorem.

Finally, in Section 5, we present a uniqueness theorem which shows that remarkably little information is required to identify an analytic function. Just as Cauchy's Integral Formula tells us that the values of an analytic function within a circle are uniquely determined by the values of the function on the circle, so the Uniqueness Theorem tells us that the values of an analytic function on a region  $\mathcal{R}$  are uniquely determined by the values of the function on a sequence of points that converges to a point of  $\mathcal{R}$ .

This unit contains several applications of the Monotone Convergence Theorem for real sequences. This theorem is about *monotonic* sequences, that is, real sequences  $(x_n)$  that are either increasing ( $x_1 \leq x_2 \leq \cdots$ ) or decreasing ( $x_1 \geq x_2 \geq \cdots$ ). We say that a real sequence  $(x_n)$  is *bounded above* if there is a real number  $K$  with  $x_n \leq K$ , for  $n = 1, 2, \dots$ , and  $(x_n)$  is *bounded below* if there is a real number  $K$  with  $x_n \geq K$ , for  $n = 1, 2, \dots$ .

We state the theorem here for convenience.

**Monotone Convergence Theorem** Any real sequence that is increasing and bounded above, or decreasing and bounded below, converges.

## Unit guide

This unit forms the basis for Unit B4 on Laurent series. The most important sections are Sections 2 and 3 – especially Section 3.

Section 1 is slightly longer than the other sections, so if you have not already studied series in the subject of real analysis, then you should be prepared to spend extra time on this part of the unit.

On the other hand, if you have studied real series before, then you will be familiar with the form of many of the results presented here. There are, however, important differences between real and complex series. For example, you will see in Section 3 that a complex function that is analytic at a point  $\alpha$  has an infinite Taylor series about  $\alpha$ , whereas, in contrast, a real function that is differentiable on an open interval containing a point  $\alpha$  may not have an infinite Taylor series about  $\alpha$ .

Section 5 contains some theoretical work on uniqueness that prepares the way for the study of *analytic continuation* in Unit C1. The final subsection indicates an alternative approach to the definition of complex functions which you may find interesting if you have the time.

## 1 Complex series

After working through this section, you should be able to:

- explain what is meant by *convergent* and *divergent* series
- use partial sums to find the *sum* of a given convergent series
- use the Non-null Test to show that a given series diverges
- recognise a convergent geometric series, and write down its sum
- use the Combination Rules for series to find the sum of a convergent series
- check the convergence of a series by inspecting its real and imaginary parts
- check for convergence using the Comparison Test
- explain what is meant by an *absolutely convergent* series
- check for absolute convergence using the Absolute Convergence Test and the Ratio Test.

## 1.1 Convergent series

Given any finite collection of complex numbers, we can form the sum of those numbers. In this section we investigate whether this idea of summation can be generalised to an infinite sequence of complex numbers  $(z_n)$ . Roughly speaking, the idea is to see whether successive sums

$$z_1, \quad z_1 + z_2, \quad z_1 + z_2 + z_3, \quad \dots$$

(called *partial sums*) approach a limit as more and more terms are included. In preparation for this, we make the following definitions.

### Definitions

Given a sequence  $(z_n)$  of complex numbers, the expression

$$z_1 + z_2 + z_3 + \dots$$

is called an **infinite series**, or simply a **series**. The number  $z_n$  is called the  **$n$ th term** of the series.

The  **$n$ th partial sum** of the series is the complex number

$$s_n = z_1 + z_2 + \dots + z_n = \sum_{k=1}^n z_k.$$

We sometimes refer to a series as a *complex series* if we wish to stress the fact that the terms of the series are complex numbers, not necessarily real. We refer to a series as a *real series* if all its terms are real numbers.

### Remarks

1. Note that in the definition of the  $n$ th partial sum, we have used the letter  $k$  as an index. We choose  $k$  rather than, say,  $n$  or  $i$ , because  $n$  is already being used for a different purpose, and, in complex analysis, we prefer to reserve  $i$  for a square root of  $-1$ .
2. Observe the difference between the sequence of *terms*  $(z_n)$  of the series and the sequence of *partial sums*  $(s_n)$  of the series.

As for finite sums, we frequently use sigma notation to represent an infinite series. Thus we write

$$\sum_{n=1}^{\infty} z_n = z_1 + z_2 + z_3 + \dots$$

If we need to begin a series with a term other than  $z_1$ , then we write, for example,

$$\sum_{n=0}^{\infty} z_n = z_0 + z_1 + z_2 + \dots \quad \text{or} \quad \sum_{n=3}^{\infty} z_n = z_3 + z_4 + z_5 + \dots$$

For such a series, the  $n$ th partial sum  $s_n$  is defined to be the sum of all the

terms up to and including  $z_n$ . For instance, the fifth partial sum of the series  $\sum_{n=3}^{\infty} z_n$  is

$$s_5 = \sum_{k=3}^5 z_k = z_3 + z_4 + z_5,$$

and the first and second partial sums are not defined.

### Exercise 1.1

Evaluate the 0th, 1st, 2nd and 3rd partial sums of the series

$$\sum_{n=0}^{\infty} \left(\frac{i}{2}\right)^n.$$

Occasionally it is possible to find an explicit expression for the  $n$ th partial sum of a series. For example, using the standard formula for the sum of a finite geometric series (Remark 2 after Theorem 1.3 of Unit A1), we can see that the  $n$ th partial sum of the series in Exercise 1.1 is given by

$$s_n = \frac{1 - (i/2)^{n+1}}{1 - (i/2)}.$$

Since  $((i/2)^n)$  is a null sequence, it follows that

$$s_n \rightarrow \frac{1}{1 - (i/2)} \text{ as } n \rightarrow \infty.$$

Thus it seems reasonable to define the *sum* of the series in Exercise 1.1 to be  $1/(1 - i/2)$ . More generally, we make the following definitions.

### Definitions

The complex series

$$z_1 + z_2 + z_3 + \cdots$$

is **convergent** with **sum**  $s$  if the sequence  $(s_n)$  of partial sums converges to  $s$ . In this case, we say that the series **converges** to  $s$ , and write

$$z_1 + z_2 + z_3 + \cdots = s \quad \text{or} \quad \sum_{n=1}^{\infty} z_n = s.$$

The series **diverges** if the sequence  $(s_n)$  diverges.

These definitions extend in the natural way to series that begin with a term other than  $z_1$ ; likewise, theorems and corollaries in this section could be stated with alternative choices of starting terms.

Let us now look at methods for proving that infinite series converge or diverge. One strategy is to apply methods for determining whether *sequences* converge or diverge to the sequence  $(s_n)$  of partial sums of a series.

### Example 1.1

For each of the following infinite series, calculate the  $n$ th partial sum  $s_n$ , and determine whether the series converges or diverges.

$$(a) \sum_{n=1}^{\infty} 2i = 2i + 2i + 2i + \cdots$$

$$(b) \sum_{n=0}^{\infty} 2\left(\frac{i}{3}\right)^n = 2 + 2\left(\frac{i}{3}\right) + 2\left(\frac{i}{3}\right)^2 + 2\left(\frac{i}{3}\right)^3 + \cdots$$

### Solution

(a) The  $n$ th partial sum of the series is

$$s_n = \sum_{k=1}^n 2i = 2ni.$$

Since the sequence  $(2ni)$  does not converge (it tends to  $\infty$ ), the series  $\sum_{n=1}^{\infty} 2i$  diverges.

(b) By the formula for the sum of a finite geometric series, we have

$$\begin{aligned} s_n &= 2\left(1 + \frac{i}{3} + \left(\frac{i}{3}\right)^2 + \cdots + \left(\frac{i}{3}\right)^n\right) \\ &= 2\left(\frac{1 - (i/3)^{n+1}}{1 - (i/3)}\right) \\ &= \frac{6}{3-i}(1 - (i/3)^{n+1}). \end{aligned}$$

Since  $((i/3)^n)$  is a null sequence, it follows that

$$\lim_{n \rightarrow \infty} s_n = \frac{6}{3-i} = \frac{6(3+i)}{10} = \frac{1}{5}(9+3i),$$

so the series converges to  $\frac{1}{5}(9+3i)$ .

### Exercise 1.2

For each of the following series, calculate the  $n$ th partial sum  $s_n$ , and determine whether the series converges or diverges. If the series converges, then calculate its sum.

$$(a) \sum_{n=1}^{\infty} 7(-i)^n \quad (b) \sum_{n=1}^{\infty} \left(\frac{1-i}{2}\right)^n$$

It is sometimes possible to show that a series converges or diverges without having to calculate its partial sums. One technique for proving divergence relies on the following theorem.

### Theorem 1.1

If  $\sum_{n=1}^{\infty} z_n$  converges, then  $(z_n)$  is a null sequence.

**Proof** Let  $s_n = \sum_{k=1}^n z_k$  be the  $n$ th partial sum of  $\sum_{n=1}^{\infty} z_n$ . Then  $(s_n)$  is a convergent sequence, with limit  $s$  (say). Since

$$s_n = s_{n-1} + z_n, \quad \text{for } n = 2, 3, \dots,$$

we have

$$z_n = s_n - s_{n-1}, \quad \text{for } n = 2, 3, \dots$$

Hence, by the Combination Rules for sequences (Theorem 1.3 of Unit A3),

$$\lim_{n \rightarrow \infty} z_n = \lim_{n \rightarrow \infty} (s_n - s_{n-1}) = \lim_{n \rightarrow \infty} s_n - \lim_{n \rightarrow \infty} s_{n-1} = s - s = 0.$$

Thus  $(z_n)$  is a null sequence. ■

Theorem 1.1 shows that a series cannot converge unless its terms form a null sequence. We therefore have the following divergence test.

### Corollary Non-null Test

If the sequence  $(z_n)$  is not null, then the series  $\sum_{n=1}^{\infty} z_n$  diverges.

For example, the series

$$\sum_{n=1}^{\infty} ni = i + 2i + 3i + \dots$$

diverges, since the sequence  $(ni)$  is not null (it tends to  $\infty$ ).

It is important to realise that the *converse* of the Non-null Test is false. That is, if  $(z_n)$  is a null sequence, then it does *not* follow that

$$\sum_{n=1}^{\infty} z_n$$

converges: it may converge, but it may diverge. For example, the sequence  $(1/n)$  is null and yet the series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

diverges, as you will see in Theorem 1.3. This series is called the **harmonic series**.

### Exercise 1.3

Use the Non-null Test to show that each of the following series diverges.

$$(a) \sum_{n=1}^{\infty} (1+i)^n \quad (b) \sum_{n=1}^{\infty} i(-1)^n \quad (c) \sum_{n=1}^{\infty} \frac{n^2 + i}{2n^2 + n + 3}$$

## 1.2 Some basic series

Many of the series we have mentioned so far have been of the form

$$\sum_{n=0}^{\infty} az^n = a + az + az^2 + \cdots$$

Such series are known as **geometric series** with **common ratio**  $z$ . The following theorem enables us to decide whether a given geometric series converges or diverges.

### Theorem 1.2 Geometric Series

Consider the series  $\sum_{n=0}^{\infty} az^n$ , where  $a, z \in \mathbb{C}$ .

- (a) If  $|z| < 1$ , then the series converges to  $a/(1-z)$ .
- (b) If  $|z| \geq 1$  and  $a \neq 0$ , then the series diverges.

### Proof

- (a) If  $z \neq 1$ , then the  $n$ th partial sum is given by

$$s_n = a + az + az^2 + \cdots + az^n = a \left( \frac{1 - z^{n+1}}{1 - z} \right).$$

Furthermore, if  $|z| < 1$ , then  $(z^n)$  is a basic null sequence (Theorem 1.2(b) of Unit A3). So, by the Combination Rules for sequences,

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} a \left( \frac{1 - z^{n+1}}{1 - z} \right) = \frac{a}{1 - z}.$$

Thus the series  $\sum_{n=0}^{\infty} az^n$  converges to  $a/(1-z)$ .

- (b) If  $|z| \geq 1$ , then  $(z^n)$  does not converge to 0 (see Theorem 1.7 of Unit A3). Since  $a \neq 0$ , the sequence  $(az^n)$  does not converge to 0 either. Hence the series diverges, by the Non-null Test. ■

**Exercise 1.4**

Use Theorem 1.2 to check your answers to Exercise 1.2.

The convergence or divergence of a given complex series can often be determined by comparing it with another series that is known to converge or diverge. As you will see later in this section, such comparisons are frequently made with geometric series. Other series that are sometimes used in this way are series of the form

$$1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots,$$

where  $p$  is real. The convergence of such series depends on the value of  $p$ .

**Theorem 1.3**

The series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots$$

converges if  $p > 1$  and diverges if  $p \leq 1$ .

For example, if  $p = 2$ , then we obtain the convergent series

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots.$$

Later on in the module, you will see that this series converges to  $\pi^2/6$ . On the other hand, if  $p = 1$ , then we obtain the divergent series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots.$$

This is the harmonic series that we mentioned earlier.

**Proof** If  $p \leq 0$ , then all the terms of the series are greater than or equal to 1, so the series diverges, by the Non-null Test. This leaves two other cases to consider:  $0 < p \leq 1$  and  $p > 1$ .

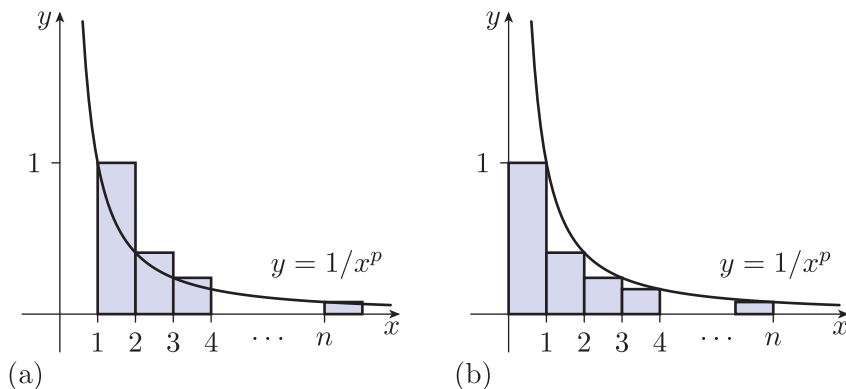
For the case where  $0 < p \leq 1$ , consider the  $n$  rectangles in the Cartesian plane that have the intervals  $[1, 2], [2, 3], \dots, [n, n+1]$  on the  $x$ -axis as their bases and have heights  $1, 1/2^p, \dots, 1/n^p$ , respectively, as shown in Figure 1.1(a). (For clarity the vertical axis in that figure has been stretched.) The  $n$ th partial sum of the series is equal to the total area of the rectangles. As the figure shows, this area is greater than the area under the graph of the real function  $f(x) = 1/x^p$  between 1 and  $n+1$ , so

$$s_n = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} > \int_1^{n+1} \frac{1}{x^p} dx.$$

Since  $1/x^p \geq 1/x$ , for  $x \geq 1$ , we can use the Monotonicity Inequality for integrals (Theorem 1.3(f) of Unit B1) to obtain

$$s_n > \int_1^{n+1} \frac{1}{x^p} dx \geq \int_1^{n+1} \frac{1}{x} dx = \log(n+1).$$

It follows that the sequence of partial sums  $(s_n)$  tends to infinity, so the series diverges for  $p \leq 1$ .



**Figure 1.1** (a) Overestimating the area under the graph of  $y = 1/x^p$ , for  $0 < p \leq 1$  (b) Underestimating the area under the graph of  $y = 1/x^p$ , for  $p > 1$

For the case where  $p > 1$ , we construct another set of  $n$  rectangles, this time with bases  $[0, 1], [1, 2], \dots, [n-1, n]$  and heights  $1, 1/2^p, \dots, 1/n^p$ , respectively, as shown in Figure 1.1(b). The  $n$ th partial sum is equal to the total area of the rectangles, and these rectangles lie *beneath* the graph of the real function  $f(x) = 1/x^p$ . If we remove the first rectangle by subtracting 1, then it follows that

$$s_n - 1 = \frac{1}{2^p} + \frac{1}{3^p} + \dots + \frac{1}{n^p} < \int_1^n \frac{1}{x^p} dx = \left[ \frac{x^{1-p}}{1-p} \right]_1^n.$$

Since  $p > 1$ , we deduce that

$$s_n - 1 < \left( \frac{n^{1-p}}{1-p} - \frac{1}{1-p} \right) = \frac{1}{p-1} \left( 1 - \frac{1}{n^{p-1}} \right) < \frac{1}{p-1},$$

so the sequence  $(s_n)$  is increasing and bounded above by  $1 + 1/(p-1)$ . By the Monotone Convergence Theorem, stated in the Introduction, the sequence  $(s_n)$  converges, and hence the series converges. ■

## 1.3 Convergence theorems

When we investigate the convergence properties of series, it is not always possible, or convenient, to work directly with partial sums. You have already seen that the Non-null Test can sometimes be used to check *divergence* without the need to calculate partial sums, so let us now turn our attention to some rules for checking *convergence*.

We begin with the Combination Rules.

**Theorem 1.4 Combination Rules for Series**

If the series  $\sum_{n=1}^{\infty} z_n$  and  $\sum_{n=1}^{\infty} w_n$  both converge, then

$$(a) \quad \text{Sum Rule} \quad \sum_{n=1}^{\infty} (z_n + w_n) = \sum_{n=1}^{\infty} z_n + \sum_{n=1}^{\infty} w_n$$

$$(b) \quad \text{Multiple Rule} \quad \sum_{n=1}^{\infty} \lambda z_n = \lambda \sum_{n=1}^{\infty} z_n, \quad \text{for } \lambda \in \mathbb{C}.$$

For example, from the formula for geometric series (Theorem 1.2), we know that

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = 1 \quad \text{and} \quad \sum_{n=1}^{\infty} \left(\frac{1-i}{2}\right)^n = -i$$

(the second sum was calculated in Exercise 1.2(b)). Hence

$$\sum_{n=1}^{\infty} \left(7\left(\frac{1}{2}\right)^n + i\left(\frac{1-i}{2}\right)^n\right) = 7 \times 1 + i \times (-i) = 8.$$

**Proof** The proofs of these Combination Rules follow from the corresponding Combination Rules for sequences. We will prove the Sum Rule, and leave you to prove the Multiple Rule in Exercise 1.5.

Let  $(s_n)$  and  $(t_n)$  be the sequences of partial sums given by

$$s_n = \sum_{k=1}^n z_k \quad \text{and} \quad t_n = \sum_{k=1}^n w_k,$$

and let

$$s = \sum_{k=1}^{\infty} z_k \quad \text{and} \quad t = \sum_{k=1}^{\infty} w_k.$$

Then  $s_n \rightarrow s$ ,  $t_n \rightarrow t$ , and, by the Sum Rule for sequences,  $s_n + t_n \rightarrow s + t$ . But

$$s_n + t_n = \sum_{k=1}^n (z_k + w_k),$$

so we see that  $\sum_{k=1}^{\infty} (z_k + w_k)$  converges to  $s + t$ . Hence

$$\sum_{k=1}^{\infty} (z_k + w_k) = s + t = \sum_{k=1}^{\infty} z_k + \sum_{k=1}^{\infty} w_k.$$



### Exercise 1.5

Prove the Multiple Rule for series.

An immediate consequence of the Combination Rules is that if  $(z_n)$  is a sequence for which both the series  $\sum_{n=1}^{\infty} \operatorname{Re} z_n$  and  $\sum_{n=1}^{\infty} \operatorname{Im} z_n$  converge, then

the series  $\sum_{n=1}^{\infty} z_n$  converges and

$$\sum_{n=1}^{\infty} z_n = \sum_{n=1}^{\infty} \operatorname{Re} z_n + i \sum_{n=1}^{\infty} \operatorname{Im} z_n.$$

In fact the converse is also true. For if the series  $\sum_{n=1}^{\infty} z_n$  converges to  $s$ , say, then

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \operatorname{Re} z_k = \lim_{n \rightarrow \infty} \operatorname{Re} \left( \sum_{k=1}^n z_k \right) = \operatorname{Re} s,$$

where we have applied Theorem 1.4(c) of Unit A3 to interchange the order of the limit and the real part function.

So  $\sum_{n=1}^{\infty} \operatorname{Re} z_n$  converges to  $\operatorname{Re} s$ , and similarly  $\sum_{n=1}^{\infty} \operatorname{Im} z_n$  converges to  $\operatorname{Im} s$ .

The series  $\sum_{n=1}^{\infty} \operatorname{Re} z_n$  and  $\sum_{n=1}^{\infty} \operatorname{Im} z_n$  are called the **real part** and

**imaginary part**, respectively, of the series  $\sum_{n=1}^{\infty} z_n$ .

We have therefore established the following equivalence between the convergence of a series and the convergence of its real and imaginary parts.

### Theorem 1.5

The series  $\sum_{n=1}^{\infty} z_n$  converges if and only if both  $\sum_{n=1}^{\infty} \operatorname{Re} z_n$  and  $\sum_{n=1}^{\infty} \operatorname{Im} z_n$  converge. In this case,

$$\sum_{n=1}^{\infty} z_n = \sum_{n=1}^{\infty} \operatorname{Re} z_n + i \sum_{n=1}^{\infty} \operatorname{Im} z_n.$$

This theorem can be used to check whether a series converges *or* diverges.

For example, the series

$$\sum_{n=1}^{\infty} \left( \frac{1}{n^2} + \frac{i}{n} \right) = (1+i) + \left( \frac{1}{4} + \frac{i}{2} \right) + \left( \frac{1}{9} + \frac{i}{3} \right) + \cdots$$

diverges because its imaginary part  $\sum_{n=1}^{\infty} 1/n$  diverges. On the other hand, the series

$$\sum_{n=1}^{\infty} \left( \frac{1}{n^2} + \frac{i}{n^3} \right) = (1+i) + \left( \frac{1}{4} + \frac{i}{8} \right) + \left( \frac{1}{9} + \frac{i}{27} \right) + \cdots$$

converges because both  $\sum_{n=1}^{\infty} 1/n^2$  and  $\sum_{n=1}^{\infty} 1/n^3$  converge.

Theorem 1.5 can also be used in a rather unexpected way to find the sums of certain *real* series.

### Example 1.2

Let  $x$  be an arbitrary real number. Find the sum of the series

$$\sum_{n=0}^{\infty} \frac{1}{2^n} \sin nx.$$

### Solution

First observe that

$$\sin nx = \operatorname{Im}(e^{inx}), \quad \text{for } n = 0, 1, 2, \dots$$

Hence, by Theorem 1.5,

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{1}{2^n} \sin nx &= \operatorname{Im} \left( \sum_{n=0}^{\infty} \frac{1}{2^n} e^{inx} \right) \\ &= \operatorname{Im} \left( \sum_{n=0}^{\infty} \left( \frac{1}{2} e^{ix} \right)^n \right) \\ &= \operatorname{Im} \left( \frac{1}{1 - \frac{1}{2} e^{ix}} \right), \end{aligned}$$

where, to obtain the last line, we have applied Theorem 1.2(a) on geometric series. Multiplying the numerator and denominator of

$$\frac{1}{1 - \frac{1}{2} e^{ix}}$$

by the complex conjugate of  $1 - \frac{1}{2} e^{ix}$ , namely

$$\overline{1 - \frac{1}{2} e^{ix}} = 1 - \frac{1}{2} \overline{e^{ix}} = 1 - \frac{1}{2} e^{-ix},$$

we obtain

$$\begin{aligned}
 \sum_{n=0}^{\infty} \frac{1}{2^n} \sin nx &= \operatorname{Im} \left( \frac{1 - \frac{1}{2}e^{-ix}}{(1 - \frac{1}{2}e^{ix})(1 - \frac{1}{2}e^{-ix})} \right) \\
 &= \operatorname{Im} \left( \frac{1 - \frac{1}{2}(\cos(-x) + i \sin(-x))}{1 - \frac{1}{2}e^{-ix} - \frac{1}{2}e^{ix} + \frac{1}{4}} \right) \\
 &= \operatorname{Im} \left( \frac{(1 - \frac{1}{2}\cos x) + \frac{1}{2}i \sin x}{1 + \frac{1}{4} - \frac{1}{2}(e^{ix} + e^{-ix})} \right) \\
 &= \frac{\frac{1}{2} \sin x}{\frac{5}{4} - \cos x} \\
 &= \frac{2 \sin x}{5 - 4 \cos x}.
 \end{aligned}$$

### Exercise 1.6

Find the sum of the series  $\sum_{n=0}^{\infty} \frac{1}{2^n} \cos nx$ , where  $x \in \mathbb{R}$ .

We are often required to prove that a given series converges without needing to know the value of the sum. In such cases it may be possible to use a theorem like the following one, which enables us to test a series for convergence by comparing it with another series that is known to converge.

### Theorem 1.6 Comparison Test

If  $\sum_{n=1}^{\infty} a_n$  is a convergent real series of non-negative terms, and

$$|z_n| \leq a_n, \quad \text{for } n = 1, 2, \dots,$$

then the series  $\sum_{n=1}^{\infty} z_n$  converges.

Before proving this theorem, we illustrate how it is used by reconsidering the series in Example 1.2. This method of establishing convergence is simpler but it does not yield a value for the sum.

### Example 1.3

Show that the following series converges, where  $x \in \mathbb{R}$ :

$$\sum_{n=0}^{\infty} \frac{1}{2^n} \sin nx.$$

**Solution**

First observe that

$$\left| \frac{1}{2^n} \sin nx \right| \leq \frac{1}{2^n}, \quad \text{for } n = 0, 1, 2, \dots$$

Now, the series  $\sum_{n=0}^{\infty} \frac{1}{2^n}$  converges, by Theorem 1.2(a), so the series

$$\sum_{n=0}^{\infty} \frac{1}{2^n} \sin nx$$

converges, by the Comparison Test.

(Although the series includes a term corresponding to  $n = 0$ , this does not affect the convergence of the series. It is therefore legitimate to use the Comparison Test here.)

**Exercise 1.7**

Show that the following series converges:

$$\sum_{n=1}^{\infty} \frac{\cos n}{n\sqrt{n}}.$$

The next exercise asks you to use the Comparison Test to prove a striking result about complex series that will be needed in the next subsection.

**Exercise 1.8**

Use the Comparison Test to prove that if the series  $\sum_{n=1}^{\infty} |z_n|$  converges, then the series  $\sum_{n=1}^{\infty} z_n$  converges.

We end this subsection by proving Theorem 1.6.

**Proof of the Comparison Test** Let  $\sum_{n=1}^{\infty} z_n$  be a complex series, and

let  $\sum_{n=1}^{\infty} a_n$  be a convergent real series of non-negative terms satisfying

$$|z_n| \leq a_n, \quad \text{for } n = 1, 2, \dots$$

We will split the proof that the series  $\sum_{n=1}^{\infty} z_n$  converges into three stages.

We prove it first for the case where all the terms  $z_n$  are real and non-negative, then for the case where they are arbitrary real numbers, and finally for the case where they are arbitrary complex numbers.

1. If all the terms  $z_n$  are real and non-negative, then  $0 \leq z_n \leq a_n$ ,

for  $n = 1, 2, \dots$ . So the partial sums  $s_n = \sum_{k=1}^n z_k$  satisfy

$$s_n \leq s_{n+1} \quad \text{and} \quad s_n \leq \sum_{k=1}^{\infty} a_k, \quad \text{for } n = 1, 2, \dots$$

Hence  $(s_n)$  is an increasing real sequence, bounded above by the sum  $\sum_{k=1}^{\infty} a_k$ . So, by the Monotone Convergence Theorem,  $(s_n)$  converges.

It follows that the series  $\sum_{n=1}^{\infty} z_n$  converges.

2. If all the terms  $z_n$  are real numbers, then we can separate the positive and negative terms by writing  $z_n = z_n^+ - z_n^-$ , where

$$z_n^+ = \begin{cases} z_n, & z_n > 0, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad z_n^- = \begin{cases} -z_n, & z_n < 0, \\ 0, & \text{otherwise.} \end{cases}$$

(For example, if

$$\sum_{n=1}^{\infty} z_n = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \dots,$$

then

$$\sum_{n=1}^{\infty} z_n^+ = 1 + 0 + \frac{1}{4} + 0 + \frac{1}{16} + \dots$$

and

$$\sum_{n=1}^{\infty} z_n^- = 0 + \frac{1}{2} + 0 + \frac{1}{8} + 0 + \dots .)$$

Then, for  $n = 1, 2, \dots$ ,

$$|z_n^+| \leq |z_n| \leq a_n \quad \text{and} \quad |z_n^-| \leq |z_n| \leq a_n,$$

so, by step 1 of the proof,

$$\sum_{n=1}^{\infty} z_n^+ \quad \text{and} \quad \sum_{n=1}^{\infty} z_n^-$$

converge. Thus, by the Combination Rules, the series

$$\sum_{n=1}^{\infty} z_n = \sum_{n=1}^{\infty} (z_n^+ - z_n^-)$$

also converges.

3. Finally, for complex terms  $z_n$  we have

$$|\operatorname{Re} z_n| \leq |z_n| \leq a_n \quad \text{and} \quad |\operatorname{Im} z_n| \leq |z_n| \leq a_n,$$

so, by step 2 of the proof, the series

$$\sum_{n=1}^{\infty} \operatorname{Re} z_n \quad \text{and} \quad \sum_{n=1}^{\infty} \operatorname{Im} z_n$$

converge. It follows from Theorem 1.5 that the series

$$\sum_{n=1}^{\infty} z_n$$

also converges, as required. ■

## 1.4 Absolute convergence

In Exercise 1.8 you saw that

$$\text{if } \sum_{n=1}^{\infty} |z_n| \text{ converges, then } \sum_{n=1}^{\infty} z_n \text{ converges.}$$

This observation is useful because it is often easier to check the convergence of the former series of real non-negative terms than it is to check the convergence of the latter series of complex terms.

### Definition

The complex series  $\sum_{n=1}^{\infty} z_n$  is **absolutely convergent** if the real series  $\sum_{n=1}^{\infty} |z_n|$  is convergent.

We also say that a series **converges absolutely** if it is absolutely convergent.

For example, if  $|z| < 1$ , then the geometric series  $\sum_{n=1}^{\infty} az^n$  is absolutely convergent. This is because

$$\sum_{n=1}^{\infty} |az^n| = \sum_{n=1}^{\infty} |a||z|^n,$$

and the series on the right is a (real) geometric series with common ratio  $|z|$  that is less than 1, so it converges.

We can now write out the result of Exercise 1.8 in the form of a convergence test.

**Theorem 1.7 Absolute Convergence Test**

If the series  $\sum_{n=1}^{\infty} z_n$  is absolutely convergent, then the series  $\sum_{n=1}^{\infty} z_n$  converges.

The converse to this theorem is *false*, because there are convergent series that do *not* converge absolutely. We will look at examples of such series shortly (in Exercise 1.10), but for now let us see how the Absolute Convergence Test can be applied.

**Example 1.4**

Show that the following series converges:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} i^n}{n^3} = i - \frac{i^2}{2^3} + \frac{i^3}{3^3} - \frac{i^4}{4^3} + \frac{i^5}{5^3} - \cdots.$$

**Solution**

The series

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1} i^n}{n^3} \right| = \sum_{n=1}^{\infty} \frac{1}{n^3}$$

converges, by Theorem 1.3. So, by the Absolute Convergence Test, the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} i^n}{n^3}$$

converges.

The next example uses the Absolute Convergence Test to generalise the result in Theorem 1.3.

**Example 1.5**

Prove that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^z} = 1 + \frac{1}{2^z} + \frac{1}{3^z} + \cdots$$

converges whenever  $\operatorname{Re} z > 1$ .

**Solution**

First recall that  $|e^\alpha| = e^{\operatorname{Re} \alpha}$  (Theorem 4.1(b) of Unit A2). Then

$$\left| \frac{1}{n^z} \right| = |e^{-z \operatorname{Log} n}| = e^{-(\operatorname{Re} z) \operatorname{Log} n} = \frac{1}{n^{\operatorname{Re} z}}.$$

But if  $\operatorname{Re} z > 1$ , then  $\sum_{n=1}^{\infty} \frac{1}{n^{\operatorname{Re} z}}$  converges, by Theorem 1.3, so  $\sum_{n=1}^{\infty} \frac{1}{n^z}$  converges absolutely. Hence

$$\sum_{n=1}^{\infty} \frac{1}{n^z}$$

converges, by the Absolute Convergence Test.

The series  $\sum_{n=1}^{\infty} \frac{1}{n^z}$ , for  $\operatorname{Re} z > 1$ , is used to define the famous zeta function, which you will meet in Unit C2.

### Exercise 1.9

Determine which of the following series are absolutely convergent.

(a)  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$

(b)  $\sum_{n=0}^{\infty} \frac{(-1)^n (1+i)^n}{2^n} = 1 - \frac{(1+i)}{2} + \frac{(1+i)^2}{4} - \frac{(1+i)^3}{8} + \frac{(1+i)^4}{16} - \dots$

The Absolute Convergence Test states that if the series  $\sum_{n=1}^{\infty} |z_n|$  converges, then so does  $\sum_{n=1}^{\infty} z_n$ , but it does not indicate any connection between the sums of these two series. The following result, which is an infinite form of the Triangle Inequality (see Theorem 5.1 of Unit A1, and its corollary), provides a connection.

### Theorem 1.8 Triangle Inequality for Series

If the series  $\sum_{n=1}^{\infty} z_n$  is absolutely convergent, then

$$\left| \sum_{n=1}^{\infty} z_n \right| \leq \sum_{n=1}^{\infty} |z_n|.$$

For example, the series  $\sum_{n=1}^{\infty} \frac{i^n}{n^2}$  is absolutely convergent because

$$\sum_{n=1}^{\infty} \left| \frac{i^n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2},$$

and this real series converges by Theorem 1.3. So it follows from Theorem 1.8 that

$$\left| \sum_{n=1}^{\infty} \frac{i^n}{n^2} \right| \leq \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

**Proof** By the Triangle Inequality for finite collections of complex numbers (see the corollary to Theorem 5.1 of Unit A1),

$$\left| \sum_{k=1}^n z_k \right| \leq \sum_{k=1}^n |z_k|, \quad \text{for } n = 1, 2, \dots$$

Now, if  $(a_n)$  and  $(b_n)$  are two convergent real sequences with  $a_n \leq b_n$ , for  $n = 1, 2, \dots$ , then it can easily be proved from the definition of a limit that  $\lim_{n \rightarrow \infty} a_n \leq \lim_{n \rightarrow \infty} b_n$ . Let us apply this fact with

$$a_n = \left| \sum_{k=1}^n z_k \right| \quad \text{and} \quad b_n = \sum_{k=1}^n |z_k|.$$

First we need to check that these sequences  $(a_n)$  and  $(b_n)$  converge.

We are given that the series

$$\sum_{n=1}^{\infty} z_n$$

is absolutely convergent, and the Absolute Convergence Test tells us that it is therefore convergent. Since the series converges absolutely, we deduce that the sequence  $(b_n)$  converges. Since the series converges, we deduce that the sequence

$$s_n = \sum_{k=1}^n z_k$$

converges. What is more,

$$\left| \sum_{k=1}^{\infty} z_k \right| = \left| \lim_{n \rightarrow \infty} \sum_{k=1}^n z_k \right| = \lim_{n \rightarrow \infty} \left| \sum_{k=1}^n z_k \right|, \quad (1.1)$$

where we have applied Theorem 1.4(a) of Unit A3 to interchange the order of the limit and the modulus function. So, in particular, the sequence  $(a_n)$  converges.

Hence  $\lim_{n \rightarrow \infty} a_n \leq \lim_{n \rightarrow \infty} b_n$ ; that is,

$$\lim_{n \rightarrow \infty} \left| \sum_{k=1}^n z_k \right| \leq \lim_{n \rightarrow \infty} \sum_{k=1}^n |z_k|. \quad (1.2)$$

Combining equations (1.1) and inequality (1.2), we obtain

$$\left| \sum_{k=1}^{\infty} z_k \right| = \lim_{n \rightarrow \infty} \left| \sum_{k=1}^n z_k \right| \leq \lim_{n \rightarrow \infty} \sum_{k=1}^n |z_k| = \sum_{k=1}^{\infty} |z_k|,$$

as required. ■

It is important to realise that not every convergent series is absolutely convergent. For example, in Exercise 1.9(a) you showed that the following series does *not* converge absolutely:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots.$$

And yet this series does converge, as we now ask you to show.

### Exercise 1.10

Use the identity

$$\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}$$

to prove that the following series converges:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots.$$

(*Hint*: Prove that both the odd and even subsequences of the sequence of partial sums converge to the same value  $\alpha$ . It then follows, by Exercise 1.14 of Unit A3, that the sequence of partial sums also converges to  $\alpha$ .)

Fortunately, many of the series we consider in this module *are* absolutely convergent, and we end this section with a test that can sometimes be used to check absolute convergence. It is based on the behaviour of the ratio between consecutive terms of a series.

### Theorem 1.9 Ratio Test

Suppose that  $\sum_{n=1}^{\infty} z_n$  is a complex series for which

$$\left| \frac{z_{n+1}}{z_n} \right| \rightarrow l \text{ as } n \rightarrow \infty.$$

- (a) If  $0 \leq l < 1$ , then  $\sum_{n=1}^{\infty} z_n$  converges absolutely (so it converges).
- (b) If  $l > 1$ , then  $\sum_{n=1}^{\infty} z_n$  diverges.

### Remarks

1. The Ratio Test yields no information if  $l = 1$ .
2. The case  $l > 1$  includes the situation where  $\left| \frac{z_{n+1}}{z_n} \right| \rightarrow \infty$  as  $n \rightarrow \infty$ .

3. In applications of the Ratio Test, it may happen that some of the terms  $z_n$  take the value 0, in which case the ratio  $z_{n+1}/z_n$  is not defined. However, if this happens for at most finitely many  $n$ , then the limit of the sequence  $(|z_{n+1}/z_n|)$  may still exist, in which case the test still applies (because the exclusion of a finite number of terms from a series does not affect whether it converges or diverges).

Before proving the Ratio Test, we illustrate how it is used.

### Example 1.6

Prove that the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n z^n}{n}$$

converges absolutely if  $|z| < 1$ , and diverges if  $|z| > 1$ .

### Solution

Let

$$z_n = \frac{(-1)^n z^n}{n}$$

(taking care to distinguish between the term  $z_n$  and the power  $z^n$ ).

The series is clearly absolutely convergent if  $z = 0$ , so let  $z \neq 0$ . Then

$$\left| \frac{z_{n+1}}{z_n} \right| = \left| \frac{(-1)^{n+1} z^{n+1}}{n+1} \right| \bigg/ \left| \frac{(-1)^n z^n}{n} \right| = \frac{n|z|}{n+1},$$

which tends to  $|z|$  as  $n \rightarrow \infty$ . It follows from the Ratio Test, applied with  $l = |z|$ , that

$$\sum_{n=1}^{\infty} \frac{(-1)^n z^n}{n}$$

is absolutely convergent if  $|z| < 1$ , and divergent if  $|z| > 1$ .

### Exercise 1.11

- (a) Use the Ratio Test to determine whether or not the series

$$\sum_{n=1}^{\infty} \frac{n^2}{3^n + i}$$

is absolutely convergent.

- (b) Prove that the series

$$\sum_{n=0}^{\infty} \frac{z^n}{n!}$$

is absolutely convergent for all  $z \in \mathbb{C}$ .

### Proof of the Ratio Test

(a) For the case  $0 \leq l < 1$ , we choose  $\varepsilon > 0$  such that

$$l + \varepsilon < 1.$$

(For example, choose  $\varepsilon = (1 - l)/2$ .) Since  $|z_{n+1}/z_n| \rightarrow l$  as  $n \rightarrow \infty$ , there is a positive integer  $N$  such that

$$\left| \frac{z_{n+1}}{z_n} \right| < l + \varepsilon, \quad \text{for all } n \geq N.$$

Thus  $z_n \neq 0$  for all  $n \geq N$ , and if  $n > N$ , then

$$\left| \frac{z_n}{z_N} \right| = \left| \frac{z_n}{z_{n-1}} \right| \left| \frac{z_{n-1}}{z_{n-2}} \right| \cdots \left| \frac{z_{N+1}}{z_N} \right| < (l + \varepsilon)^{n-N},$$

since each of the modulus terms in the middle is less than  $l + \varepsilon$ . Hence

$$|z_n| \leq |z_N|(l + \varepsilon)^{n-N}, \quad \text{for all } n \geq N. \quad (1.3)$$

Now

$$\sum_{n=N}^{\infty} |z_N|(l + \varepsilon)^{n-N} = |z_N| + |z_N|(l + \varepsilon) + |z_N|(l + \varepsilon)^2 + \cdots$$

is a geometric series with common ratio  $l + \varepsilon$ . Since  $0 < l + \varepsilon < 1$ , this series converges, so, by inequality (1.3) and the Comparison Test,

$\sum_{n=N}^{\infty} |z_n|$  also converges. Hence the series  $\sum_{n=1}^{\infty} z_n$  is absolutely

convergent, because the inclusion of a finite number of extra terms at the beginning of a series does not affect its convergence.

(b) If  $\left| \frac{z_{n+1}}{z_n} \right| \rightarrow l$  as  $n \rightarrow \infty$ , and  $l > 1$ , then there is a positive integer  $N$  such that

$$\left| \frac{z_{n+1}}{z_n} \right| > 1, \quad \text{for all } n \geq N.$$

(This also holds if  $|z_{n+1}/z_n| \rightarrow \infty$  as  $n \rightarrow \infty$ .) Thus, for all  $n > N$ , we have

$$\left| \frac{z_n}{z_N} \right| = \left| \frac{z_n}{z_{n-1}} \right| \left| \frac{z_{n-1}}{z_{n-2}} \right| \cdots \left| \frac{z_{N+1}}{z_N} \right| > 1,$$

since each of the modulus terms is greater than 1. Hence

$$|z_n| > |z_N| > 0, \quad \text{for all } n > N,$$

so the sequence  $(z_n)$  cannot converge to 0 as  $n \rightarrow \infty$ . Thus, by the

Non-null Test, the series  $\sum_{n=1}^{\infty} z_n$  diverges. ■

## Further exercises

### Exercise 1.12

For each of the following series, calculate the 0th, 1st, 2nd, 3rd and  $n$ th partial sums.

$$(a) \sum_{n=0}^{\infty} i \quad (b) \sum_{n=0}^{\infty} \frac{i}{10^n} \quad (c) \sum_{n=0}^{\infty} i^n$$

### Exercise 1.13

Determine whether each of the following series converges or diverges. If the series converges, then give its sum.

$$(a) \sum_{n=0}^{\infty} \frac{1}{\sqrt{2}} (1-i)^n \quad (b) \sum_{n=0}^{\infty} \left( \frac{1+i}{2} \right)^n \quad (c) \sum_{n=0}^{\infty} \left( \frac{1-i}{\sqrt{2}} \right)^n$$

$$(d) \sum_{n=2}^{\infty} \binom{n}{2} i^n$$

### Exercise 1.14

Use Theorem 1.5 and the result of Exercise 1.13(b) to show that

$$\sum_{n=0}^{\infty} 2^{-n/2} \cos \frac{n\pi}{4} = 1.$$

### Exercise 1.15

Determine which of the following series are absolutely convergent.

$$(a) \sum_{n=1}^{\infty} \frac{1}{n^2 + 1} \quad (b) \sum_{n=1}^{\infty} \frac{1}{n^2 + i} \quad (c) \sum_{n=1}^{\infty} \frac{2^n - i}{n^2} \quad (d) \sum_{n=1}^{\infty} \frac{i^n}{n\sqrt{n}}$$

$$(e) \sum_{n=0}^{\infty} e^{n(i-1)}$$

## 2 Power series

After working through this section, you should be able to:

- explain the terms *radius of convergence* and *disc of convergence*
- state the Radius of Convergence Theorem
- state and use the Radius of Convergence Formula
- state and use the Differentiation and Integration Rules for power series.

## 2.1 The radius of convergence

As you know, a polynomial expression in  $z$  is a finite sum of multiples of positive powers of  $z$ . The theory of series enables us to take a step beyond this definition and examine expressions that include infinitely many positive powers of  $z$ , such as

$$1 + 2z + 4z^2 + 9z^3 + \dots$$

Such series are known as *power series*. Each value of  $z$  gives rise to a different series. In the next section we will show that any analytic function can be represented by a power series; our aim here is to study power series in their own right.

### Definitions

Let  $z \in \mathbb{C}$ . An expression of the form

$$\sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \dots,$$

where  $a_n \in \mathbb{C}$ ,  $n = 0, 1, 2, \dots$ , is called a **power series about 0**.

More generally, if  $\alpha \in \mathbb{C}$ , then an expression of the form

$$\sum_{n=0}^{\infty} a_n (z - \alpha)^n = a_0 + a_1 (z - \alpha) + a_2 (z - \alpha)^2 + \dots,$$

where  $a_n \in \mathbb{C}$ ,  $n = 0, 1, 2, \dots$ , is called a **power series about  $\alpha$** .

### Remarks

1. We interpret  $(z - \alpha)^0$  in the power series about  $\alpha$  as 1, even when  $z = \alpha$ .
2. Power series often appear in disguise; for example, since

$$\sum_{n=0}^{\infty} (i - 2z)^n = \sum_{n=0}^{\infty} (-1)^n 2^n (z - \tfrac{1}{2}i)^n,$$

this geometric series is a power series about  $\frac{1}{2}i$ .

The  $z$  in a power series is a ‘variable’, like the  $z$  in the rule  $z \mapsto f(z)$  for specifying a function. Different values of  $z$  in the power series

$$\sum_{n=0}^{\infty} a_n (z - \alpha)^n$$

give different series. For some values of  $z$  the series converges; for others it diverges. We say that a power series **converges on a set  $S$**  if, for each  $z \in S$ , the corresponding series converges. For example, by Theorem 1.2, the power series

$$1 - z + z^2 - z^3 + \dots$$

converges for  $|z| < 1$  and diverges for  $|z| \geq 1$ . Hence this power series converges on the open disc  $D = \{z : |z| < 1\}$ , and indeed on any subset of  $D$ . Therefore we can define a function  $f$  with domain  $D$  and rule

$$f(z) = 1 - z + z^2 - z^3 + \cdots.$$

Also, from Theorem 1.2, the power series  $1 - z + z^2 - z^3 + \cdots$  converges to  $1/(1+z)$ , for  $|z| < 1$ . Thus an equivalent definition of  $f$  is

$$f(z) = \frac{1}{1+z} \quad (z \in D).$$

In general, if

$$A = \left\{ z : \sum_{n=0}^{\infty} a_n(z - \alpha)^n \text{ converges} \right\},$$

then we can define a function

$$f(z) = \sum_{n=0}^{\infty} a_n(z - \alpha)^n,$$

which has domain  $A$ . This accords with our usual convention that the domain of a function specified just by its rule is the set of all complex numbers to which the rule is applicable.

It is sometimes useful to refer to  $f$  as the **sum function** of the power series.

If we replace  $z$  by  $z - 1$  in the power series

$$1 - z + z^2 - z^3 + \cdots,$$

then we obtain a new power series about the point 1:

$$1 - (z - 1) + (z - 1)^2 - (z - 1)^3 + \cdots.$$

When  $|z - 1| < 1$ , this series converges to

$$\frac{1}{1 + (z - 1)} = \frac{1}{z},$$

and when  $|z - 1| > 1$ , it diverges. Thus the sum function of the power series  $1 - (z - 1) + (z - 1)^2 - \cdots$  is

$$g(z) = \frac{1}{z} \quad (z \in B),$$

where  $B$  is the open disc  $\{z : |z - 1| < 1\}$ .

It is not always possible to find a simple way of expressing the sum function of a power series as we have done here.

One of the remarkable features of power series is that the sum functions that they define always turn out to have ‘disc-shaped’ domains. This is a consequence of the following theorem, which we prove at the end of this subsection.

### Theorem 2.1 Radius of Convergence Theorem

For a given power series

$$\sum_{n=0}^{\infty} a_n(z - \alpha)^n = a_0 + a_1(z - \alpha) + a_2(z - \alpha)^2 + \cdots,$$

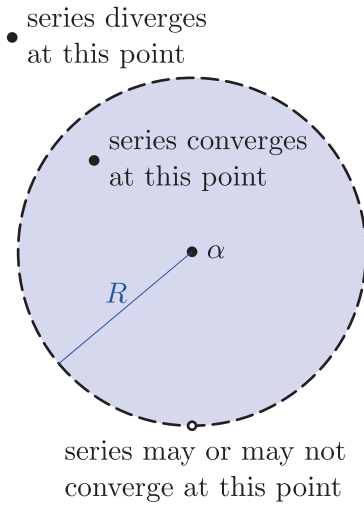
precisely one of the following possibilities occurs:

- (a) the series converges only for  $z = \alpha$
- (b) the series converges for all  $z$
- (c) there is a number  $R > 0$  such that

$$\sum_{n=0}^{\infty} a_n(z - \alpha)^n \text{ converges (absolutely) if } |z - \alpha| < R,$$

and

$$\sum_{n=0}^{\infty} a_n(z - \alpha)^n \text{ diverges if } |z - \alpha| > R.$$



**Figure 2.1** Behaviour of a power series at points inside, outside and on the circle  $|z - \alpha| = R$

Case (c) of the theorem is illustrated in Figure 2.1. The series converges at all points inside the circle of radius  $R$  centred at  $\alpha$ , and diverges at all points outside the circle. The series may or may not converge at points on the circle  $|z - \alpha| = R$  itself. We discuss the issue of convergence on the boundary circle later in the section.

#### Definition

The **radius of convergence** of a power series satisfying case (c) from the Radius of Convergence Theorem is the number  $R$ .

We extend this definition of the radius of convergence  $R$  by writing  $R = 0$  for case (a), and  $R = \infty$  for case (b).

Thus  $R$  is either a non-negative real number or it is the symbol  $\infty$ . The use of this convenient notation does not mean that we regard  $\infty$  as a real number!

Here are some examples to illustrate that all three of the cases mentioned in the theorem can occur.

#### Example 2.1

Find the radius of convergence of each of the following power series about 0.

(a)  $\sum_{n=0}^{\infty} n^n z^n$       (b)  $\sum_{n=0}^{\infty} \frac{z^n}{n!}$       (c)  $\sum_{n=0}^{\infty} z^n$

**Solution**

- (a) Clearly, the power series converges for  $z = 0$ . But for any other value of  $z$ , the power series diverges, by the Non-null Test. Indeed, given a non-zero complex number  $z$ , we see that if  $n > 1/|z|$ , then  $|n^n z^n| = (n|z|)^n > 1$ , so  $(n^n z^n)$  is not a null sequence. Thus the series converges only for  $z = 0$ , so  $R = 0$ .
- (b) In Exercise 1.11(b) you used the Ratio Test to show that this series converges for all  $z \in \mathbb{C}$ . Thus  $R = \infty$ .
- (c) This is a geometric series, so, by Theorem 1.2, it converges to  $1/(1 - z)$  for  $|z| < 1$ , and diverges for  $|z| > 1$ . Thus  $R = 1$ .

**Exercise 2.1**

- (a) Find the radius of convergence of the power series

$$\sum_{n=0}^{\infty} (4z)^n.$$

- (b) More generally, find the radius of convergence of the power series

$$\sum_{n=0}^{\infty} (\alpha z)^n$$

when  $\alpha \neq 0$ .

All the convergence tests in Section 1 can be applied to power series, since, for each value of  $z$ , a power series is just a series. The Ratio Test is particularly useful in this respect, because it can often be used to find the radius of convergence of a given power series.

**Theorem 2.2 Radius of Convergence Formula**

The power series

$$\sum_{n=0}^{\infty} a_n (z - \alpha)^n$$

has radius of convergence

$$R = \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right|,$$

provided that this limit exists (or is  $\infty$ ).

### Remarks

1. Note that the limit involves the ratio  $a_n/a_{n+1}$ , not  $a_{n+1}/a_n$ .
2. The final statement of the theorem involves the possibility that

$$\lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right| = \infty.$$

We consider this statement to mean that the sequence  $|a_n/a_{n+1}|$ ,  $n = 1, 2, \dots$ , tends to  $\infty$ . Usually we avoid the notation  $\lim_{n \rightarrow \infty} z_n = \infty$ , since this might suggest that  $\infty$  is a complex number; however, we allow this notation when applying the Radius of Convergence Formula as it is particularly convenient, and should not cause confusion in this case.

For some power series, the sequence  $(|a_n/a_{n+1}|)$  does *not* tend to a limit or to  $\infty$ , in which case the theorem does not apply. For example, in the following power series the ratio of successive coefficients does not tend to a limit or to  $\infty$ ; instead, the ratios oscillate between 2 and  $1/2$ :

$$2 + z + 2z^2 + z^3 + 2z^4 + \dots$$

Fortunately, however, the Radius of Convergence Formula can be applied for many of the power series that we will need to consider.

**Proof of the Radius of Convergence Formula** The power series converges for  $z = \alpha$ , so let us assume that  $z \neq \alpha$ . We can also assume that the sequence  $(|a_n/a_{n+1}|)$  converges to some limit  $R$ , say, which could be any non-negative number or  $\infty$ . The ratio of the  $(n+1)$ th and  $n$ th terms of the power series is

$$\frac{|a_{n+1}(z - \alpha)^{n+1}|}{|a_n(z - \alpha)^n|} = \left| \frac{a_{n+1}}{a_n} \right| |z - \alpha|,$$

which tends to  $|z - \alpha|/R$  as  $n \rightarrow \infty$ , where we interpret this limit as 0 if  $R = \infty$  and as  $\infty$  if  $R = 0$ . By the Ratio Test, the power series converges if  $|z - \alpha|/R < 1$  and diverges if  $|z - \alpha|/R > 1$ . Since

$$\frac{|z - \alpha|}{R} < 1 \iff |z - \alpha| < R$$

and

$$\frac{|z - \alpha|}{R} > 1 \iff |z - \alpha| > R,$$

we see that the radius of convergence of the power series is  $R$ . ■

### Example 2.2

Find the radius of convergence of each of the following power series.

$$(a) \sum_{n=0}^{\infty} n 2^n (z - 1)^n \quad (b) \sum_{n=0}^{\infty} \frac{(z + 2i)^n}{n!}$$

**Solution**

- (a) By the Radius of Convergence Formula (Theorem 2.2), the radius of convergence is

$$\begin{aligned} R &= \lim_{n \rightarrow \infty} \frac{n2^n}{(n+1)2^{n+1}} \\ &= \lim_{n \rightarrow \infty} \frac{n}{2(n+1)} \\ &= \lim_{n \rightarrow \infty} \frac{1}{2(1+1/n)} = \frac{1}{2}. \end{aligned}$$

- (b) By the Radius of Convergence Formula, the radius of convergence is

$$\begin{aligned} R &= \lim_{n \rightarrow \infty} \frac{1/n!}{1/(n+1)!} \\ &= \lim_{n \rightarrow \infty} \frac{(n+1)!}{n!} \\ &= \lim_{n \rightarrow \infty} (n+1) = \infty. \end{aligned}$$

**Exercise 2.2**

Determine the radius of convergence of each of the following power series.

(a)  $\sum_{n=0}^{\infty} (2^n + 4^n)z^n$       (b)  $\sum_{n=0}^{\infty} \frac{(2n)!}{n!}(z+7)^n$       (c)  $\sum_{n=0}^{\infty} (n+2^{-n})(z-1)^n$

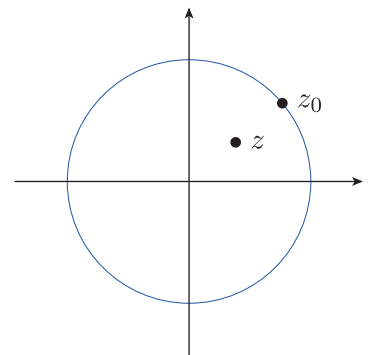
We end this subsection with a proof of the Radius of Convergence Theorem (Theorem 2.1). You may wish to omit the proof on a first reading.

**Proof of the Radius of Convergence Theorem** For simplicity let  $\alpha = 0$ , so the series is of the form

$$\sum_{n=0}^{\infty} a_n z^n.$$

The general version of the theorem follows from this special case on replacing  $z$  by  $z - \alpha$ .

The proof depends on the claim that if a power series converges at some point  $z_0$  on a circle centred at the origin, then it is *absolutely* convergent at all points  $z$  *within* the circle (see Figure 2.2). More formally, we claim that if  $z_0 \neq 0$  and the series  $\sum_{n=0}^{\infty} a_n z_0^n$  converges, then the power series  $\sum_{n=0}^{\infty} a_n z^n$  is absolutely convergent for  $|z| < |z_0|$ .



**Figure 2.2** Points  $z_0$  and  $z$  on and inside a circle

To prove this claim notice that, by Theorem 1.1,

$$\text{if } \sum_{n=0}^{\infty} a_n z_0^n \text{ converges, then } \lim_{n \rightarrow \infty} a_n z_0^n = 0,$$

so, for some constant  $K$ ,

$$|a_n z_0^n| \leq K, \quad \text{for } n = 0, 1, 2, \dots$$

(see Lemma 1.2 of Unit A3). Hence, for  $z \in \mathbb{C}$ ,

$$|a_n z^n| = |a_n z_0^n| \left| \frac{z}{z_0} \right|^n \leq K \left| \frac{z}{z_0} \right|^n, \quad \text{for } n = 0, 1, 2, \dots$$

Now, if  $|z| < |z_0|$ , then  $|z/z_0| < 1$ , so the series

$$\sum_{n=0}^{\infty} K \left| \frac{z}{z_0} \right|^n$$

converges. Hence, by the Comparison Test,

$$\sum_{n=0}^{\infty} |a_n z^n|$$

converges, as required by the claim.

Having proved the claim, let us return to the main part of the proof.

Suppose that neither case (a) nor case (b) of the theorem holds; that is, the series does not converge at  $z = 0$  only, and nor does it converge for all values of  $z$ . It follows that we can find non-zero complex numbers  $z_0$  and  $z_1$  such that

$$\sum_{n=0}^{\infty} a_n z_0^n \text{ converges} \quad \text{and} \quad \sum_{n=0}^{\infty} a_n z_1^n \text{ diverges.}$$

Now consider the set

$$E = \left\{ r \in [0, \infty) : \sum_{n=0}^{\infty} |a_n| r^n \text{ converges} \right\}.$$

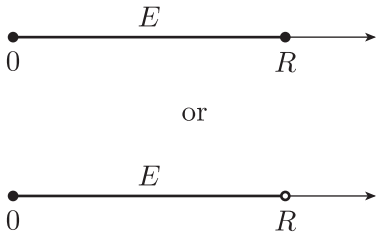
It follows from the claim, applied with our choice of  $z_0$ , that the set  $E - \{0\}$  is non-empty. Moreover,  $E$  is an interval since if  $r_0 \in E$  and  $0 < r < r_0$ , then  $r \in E$  (again by the claim).

Next we apply the Absolute Convergence Test to see that

$$\sum_{n=0}^{\infty} |a_n| |z_1|^n \text{ diverges.}$$

Thus  $|z_1| \notin E$ , so the interval  $E$  must have a finite, non-zero, right-hand endpoint,  $R$  say, which may or may not lie in  $E$  (see Figure 2.3).

If  $|z| < R$ , then  $\sum_{n=0}^{\infty} |a_n| |z|^n$  converges and hence  $\sum_{n=0}^{\infty} a_n z^n$  is absolutely convergent, as required.



**Figure 2.3** Intervals  $[0, R]$  and  $[0, R)$

On the other hand, if  $|z| > R$ , then we can choose  $r$  such that  $R < r < |z|$ ; thus  $\sum_{n=0}^{\infty} |a_n| r^n$  diverges (since  $r \notin E$ ) and hence  $\sum_{n=0}^{\infty} a_n z^n$  diverges (by the claim).

Thus case (c) holds, so the proof is complete. ■

## 2.2 The disc of convergence

According to the Radius of Convergence Theorem, if  $R$  is the radius of convergence of a power series centred at  $\alpha$ , then the series converges absolutely at all points of the open disc  $\{z : |z - \alpha| < R\}$ . This disc is called the *disc of convergence* of the power series.

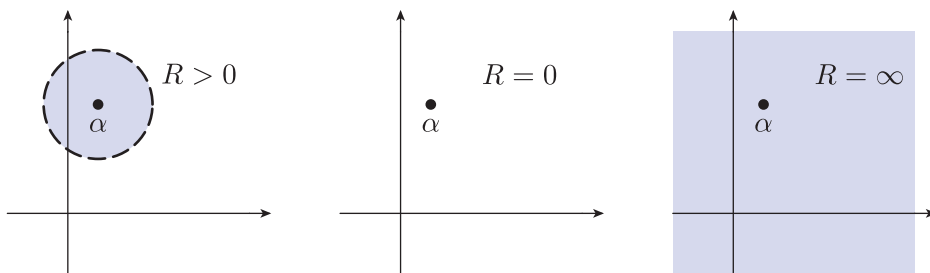
### Definition

Let  $R$  be the radius of convergence of the power series

$$\sum_{n=0}^{\infty} a_n (z - \alpha)^n.$$

Then the **disc of convergence** of the power series is the open disc  $\{z : |z - \alpha| < R\}$ . The disc of convergence is interpreted to be the empty set  $\emptyset$  if  $R = 0$ , and to be  $\mathbb{C}$  if  $R = \infty$ .

Figure 2.4 illustrates the three cases.



**Figure 2.4** Discs of convergence

### Exercise 2.3

Write down the disc of convergence for each of the power series in Example 2.1 and Exercise 2.2.

It is important to notice that the disc of convergence is an *open* disc and it may not include *all* the points at which the power series converges. This is because the series may or may not converge at points on the boundary of the disc. Unfortunately, the Radius of Convergence Theorem says nothing about what happens on the boundary. In order to gain insight into how

power series can behave on the boundary of the disc of convergence, we consider three power series each with radius of convergence 1:

$$\sum_{n=0}^{\infty} z^n, \quad \sum_{n=1}^{\infty} z^n/n^2, \quad \sum_{n=1}^{\infty} z^n/n.$$

We established that the first of these power series has radius of convergence 1 in Example 2.1(c). The next exercise asks you to show that the other two power series also have radius of convergence 1.

### Exercise 2.4

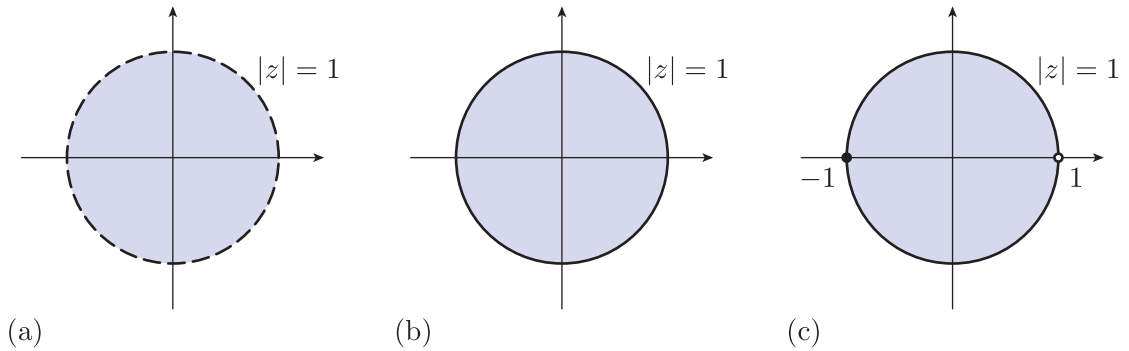
Show that each of the following power series has radius of convergence 1.

$$(a) \sum_{n=1}^{\infty} z^n/n^2 \quad (b) \sum_{n=1}^{\infty} z^n/n$$

Some power series converge only on their disc of convergence. For example, the power series

$$\sum_{n=0}^{\infty} z^n$$

is a geometric series, so it diverges at every point of the boundary circle  $\{z : |z| = 1\}$ , by Theorem 1.2(b) (see Figure 2.5(a)).



**Figure 2.5** (a)  $\sum_{n=0}^{\infty} z^n$  diverges on  $\{z : |z| = 1\}$  (b)  $\sum_{n=0}^{\infty} z^n/n^2$  converges on  $\{z : |z| = 1\}$   
(c)  $\sum_{n=0}^{\infty} z^n/n$  diverges at 1 and converges elsewhere on  $\{z : |z| = 1\}$

At the other extreme, there are power series that converge at every point on the boundary. For example, the power series

$$\sum_{n=1}^{\infty} z^n/n^2$$

is (absolutely) convergent at every point of the circle  $\{z : |z| = 1\}$  (see Figure 2.5(b)). This follows from the Absolute Convergence Test

(Theorem 1.7), because  $\sum_{n=1}^{\infty} 1/n^2$  converges.

Between these two extremes, it is possible for a power series to converge at some points on the boundary of the disc of convergence and diverge at others, as illustrated by the power series

$$\sum_{n=1}^{\infty} z^n/n.$$

If  $z = 1$ , then we obtain the divergent series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \quad (\text{by Theorem 1.3 with } p = 1),$$

but if  $z = -1$ , then we obtain the convergent series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \quad (\text{see Exercise 1.10}).$$

In fact, it can be shown (although we do not do so) that this power series converges at each point of the circle  $\{z : |z| = 1\}$ , except 1 (see Figure 2.5(c)).

These examples demonstrate the following observation.

A power series may converge at none, some, or all of the points on the boundary of its disc of convergence.

You may wonder why we did not include such points in our definition of disc of convergence. We have deliberately chosen not to do so because the analytic properties of power series are best studied on *regions* of the complex plane, such as open discs. The disc of convergence is the largest *region* on which the power series converges.

## 2.3 Differentiation of power series

You have seen that the geometric series

$$\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + z^3 + z^4 + \cdots$$

defines the function

$$f(z) = \frac{1}{1-z} \quad (|z| < 1).$$

If we differentiate this geometric series term by term, as if we were differentiating a polynomial, then we obtain a new power series:

$$\sum_{n=1}^{\infty} n z^{n-1} = 1 + 2z + 3z^2 + 4z^3 + 5z^4 + \cdots.$$

It is natural to ask whether this new series converges to the derivative

$$f'(z) = \frac{1}{(1-z)^2}.$$

The following theorem shows that it does.

**Theorem 2.3 Differentiation Rule for Power Series**

The power series

$$\sum_{n=0}^{\infty} a_n(z - \alpha)^n \quad \text{and} \quad \sum_{n=1}^{\infty} n a_n(z - \alpha)^{n-1}$$

have the same radius of convergence  $R$ . Furthermore, if

$$f(z) = \sum_{n=0}^{\infty} a_n(z - \alpha)^n,$$

then  $f$  is analytic on the disc of convergence  $\{z : |z - \alpha| < R\}$ , and

$$f'(z) = \sum_{n=1}^{\infty} n a_n(z - \alpha)^{n-1}, \quad \text{for } |z - \alpha| < R.$$

Notice that the series for  $f'$  does not contain an  $n = 0$  term. This is analogous to the loss of the constant term when a polynomial is differentiated.

The Differentiation Rule is an important theoretical tool that will be used in the next section to prove Taylor's Theorem. More practically, the rule can be used to find new power series from old ones.

**Example 2.3**

Use the Differentiation Rule and the geometric series

$$1 - z + z^2 - z^3 + z^4 - \dots \tag{2.1}$$

to prove that

$$\frac{1}{(1+z)^2} = 1 - 2z + 3z^2 - 4z^3 + \dots, \quad \text{for } |z| < 1.$$

**Solution**

The geometric series (2.1) has radius of convergence 1, so, by the Differentiation Rule, the power series

$$-1 + 2z - 3z^2 + 4z^3 - \dots,$$

obtained by differentiating (2.1) term by term, has radius of convergence 1.

Let  $f(z) = 1 - z + z^2 - z^3 + z^4 - \dots$ . Then, by Theorem 1.2(a),

$$f(z) = \frac{1}{1+z}, \quad \text{for } |z| < 1,$$

so

$$f'(z) = -\frac{1}{(1+z)^2}, \quad \text{for } |z| < 1.$$

By the Differentiation Rule,

$$f'(z) = -1 + 2z - 3z^2 + 4z^3 - \cdots, \quad \text{for } |z| < 1.$$

Hence

$$\frac{1}{(1+z)^2} = 1 - 2z + 3z^2 - 4z^3 + \cdots, \quad \text{for } |z| < 1.$$

You will be given an opportunity to practise using the Differentiation Rule at the end of this subsection. For now, let us turn our attention to a proof of the rule. The proof is challenging, so you may wish to skim through it if you are short of time.

**Proof of the Differentiation Rule for Power Series** The proof is in two steps. Again, for simplicity, we take  $\alpha = 0$ .

1. Let the power series  $\sum_{n=0}^{\infty} a_n z^n$  and  $\sum_{n=1}^{\infty} n a_n z^{n-1}$  have radii of convergence  $R$  and  $R'$ , respectively. We will prove that  $R = R'$ .

We first prove that  $R \leq R'$  by showing that if  $|z| < R$ , then the power series  $\sum_{n=1}^{\infty} n a_n z^{n-1}$  converges. To prove this, let  $r$  be a real number such that  $|z| < r < R$ . Then the series  $\sum_{n=0}^{\infty} a_n r^n$  converges, so  $\lim_{n \rightarrow \infty} a_n r^n = 0$ .

Thus there is a number  $K$  such that

$$|a_n r^n| \leq K, \quad \text{for } n = 1, 2, \dots$$

(by Lemma 1.2 of Unit A3). We now write

$$\begin{aligned} |n a_n z^{n-1}| &= n |a_n r^n| \frac{|z|^{n-1}}{r^n} \\ &= n r^{-1} |a_n r^n| \left| \frac{z}{r} \right|^{n-1} \\ &\leq n r^{-1} K \left| \frac{z}{r} \right|^{n-1}, \quad \text{for } n = 1, 2, \dots \end{aligned}$$

Since  $|z| < r$ , it follows that  $|z/r| < 1$ , so the series  $\sum_{n=1}^{\infty} n r^{-1} K \left| \frac{z}{r} \right|^{n-1}$  converges, by the Ratio Test, because

$$\lim_{n \rightarrow \infty} \frac{(n+1)r^{-1}K|z/r|^n}{nr^{-1}K|z/r|^{n-1}} = \left| \frac{z}{r} \right| < 1.$$

Hence, by the Comparison Test, the power series  $\sum_{n=1}^{\infty} n a_n z^{n-1}$  converges absolutely, as required. Thus  $R \leq R'$ .

We now prove that  $R \geq R'$  by showing that if  $|z| < R'$ , then the power series  $\sum_{n=1}^{\infty} a_n z^n$  converges. Since  $|z| < R'$ ,  $\sum_{n=1}^{\infty} |n a_n z^{n-1}|$  converges and,

by the Multiple Rule, so does  $\sum_{n=1}^{\infty} |na_n z^n|$  (we just multiply by  $|z|$ ). But

$$|a_n z^n| \leq |na_n z^n|, \quad \text{for } n = 1, 2, \dots$$

So, by the Comparison Test, the power series  $\sum_{n=1}^{\infty} a_n z^n$  is absolutely convergent, as required. Thus  $R \geq R'$ .

It follows that  $R = R'$ : the two radii of convergence are equal.

2. Next we show that  $f$  is analytic on  $D = \{z : |z| < R\}$ , and

$f'(z) = \sum_{n=1}^{\infty} na_n z^{n-1}$ , for  $|z| < R$ . To do this, let  $z$  and  $z_0$  be arbitrary distinct points in the disc  $D$  (see Figure 2.6). Then, by the Combination Rules, and using the Geometric Series Identity

$$z^n - z_0^n = (z - z_0)(z^{n-1} + z^{n-2}z_0 + \dots + z_0^{n-1})$$

(Theorem 1.3(b) of Unit A1), we see that

$$\begin{aligned} & \left| \frac{f(z) - f(z_0)}{z - z_0} - \sum_{n=1}^{\infty} na_n z_0^{n-1} \right| \\ &= \left| \sum_{n=1}^{\infty} a_n \left( \frac{z^n - z_0^n}{z - z_0} - n z_0^{n-1} \right) \right| \\ &= \left| \sum_{n=1}^{\infty} a_n (z^{n-1} + z^{n-2}z_0 + \dots + z_0^{n-1} - n z_0^{n-1}) \right| \\ &\leq |p_N(z)| + |q_N(z)|, \end{aligned}$$

where, for each  $N$ ,  $p_N$  is the polynomial function

$$p_N(z) = \sum_{n=1}^N a_n (z^{n-1} + z^{n-2}z_0 + \dots + z_0^{n-1} - n z_0^{n-1})$$

and  $q_N$  is the infinite series

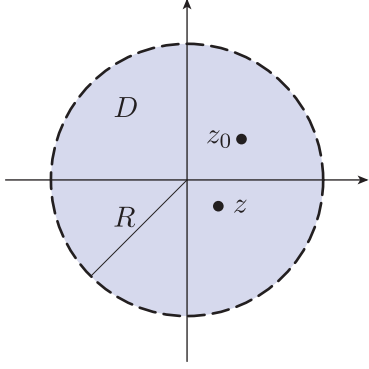
$$q_N(z) = \sum_{n=N+1}^{\infty} a_n (z^{n-1} + z^{n-2}z_0 + \dots + z_0^{n-1} - n z_0^{n-1}).$$

To prove that  $f'(z_0) = \sum_{n=1}^{\infty} na_n z_0^{n-1}$ , it is sufficient to show that, for each  $\varepsilon > 0$ , there is a  $\delta > 0$  and an integer  $N$  such that

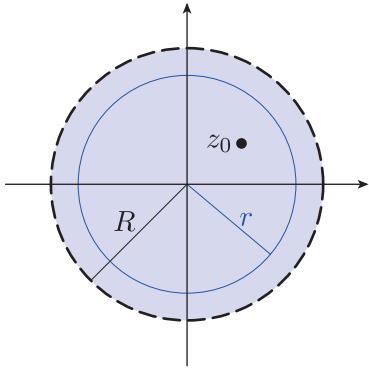
$$|z - z_0| < \delta \implies |p_N(z)| < \varepsilon/2 \text{ and } |q_N(z)| < \varepsilon/2.$$

To ensure that  $|q_N(z)| < \varepsilon/2$ , let  $r$  be a real number between  $|z_0|$  and  $R$  (see Figure 2.7), and consider the series  $\sum_{n=1}^{\infty} na_n r^{n-1}$ . Since this series is absolutely convergent, with value  $l$ , say, we can choose  $N$  so that

$$\sum_{n=N+1}^{\infty} n|a_n| r^{n-1} = \left| l - \sum_{n=1}^N n|a_n| r^{n-1} \right| < \varepsilon/4.$$



**Figure 2.6** Points  $z$  and  $z_0$  in the disc  $D$



**Figure 2.7** Circle of radius  $r$ , where  $|z_0| < r < R$

Now take  $\delta = r - |z_0|$ . If  $|z - z_0| < \delta$  (see Figure 2.8), then, on writing  $z = (z - z_0) + z_0$ , we see from the Triangle Inequality that

$$|z| \leq |z - z_0| + |z_0| < \delta + |z_0| = (r - |z_0|) + |z_0| = r.$$

So  $|z| < r$ . Furthermore, since  $|z_0| < r$ , we have

$$\begin{aligned} |q_N(z)| &= \left| \sum_{n=N+1}^{\infty} a_n(z^{n-1} + z^{n-2}z_0 + \cdots + z_0^{n-1} - nz_0^{n-1}) \right| \\ &\leq \sum_{n=N+1}^{\infty} |a_n(z^{n-1} + z^{n-2}z_0 + \cdots + z_0^{n-1} - nz_0^{n-1})|, \end{aligned}$$

by the Triangle Inequality for Series (Theorem 1.8), which we can apply provided that it can be shown that this final series converges. In fact, this series does converge, by the Comparison Test (Theorem 1.6), because

$$\begin{aligned} &|a_n(z^{n-1} + z^{n-2}z_0 + \cdots + z_0^{n-1} - nz_0^{n-1})| \\ &\leq |a_n|(|z^{n-1}| + |z^{n-2}z_0| + \cdots + |z_0^{n-1}| + n|z_0^{n-1}|) \\ &< |a_n|r^{n-1}(1 + 1 + \cdots + 1 + n) \\ &= 2n|a_n|r^{n-1} \end{aligned}$$

and

$$2 \sum_{n=N+1}^{\infty} n|a_n|r^{n-1} < 2(\varepsilon/4) = \varepsilon/2,$$

by our choice of  $N$ . Hence

$$|q_N(z)| \leq 2 \sum_{n=N+1}^{\infty} n|a_n|r^{n-1} < \varepsilon/2.$$

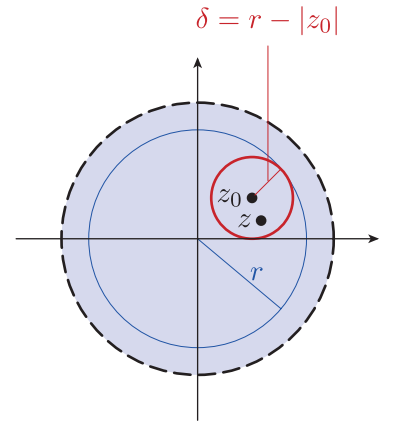
Finally, to ensure that  $|p_N(z)| < \varepsilon/2$ , notice that  $p_N$  is a polynomial function such that  $p_N(z_0) = 0$ . Since polynomial functions are continuous, we can, if necessary, make  $\delta$  even smaller than  $r - |z_0|$ , so that

$$|p_N(z)| = |p_N(z) - p_N(z_0)| < \varepsilon/2, \quad \text{for } |z - z_0| < \delta.$$

This completes the proof that  $f'(z_0) = \sum_{n=1}^{\infty} na_n z_0^{n-1}$  because, given any

$\varepsilon > 0$ , we have shown how to find  $\delta$  and  $N$  such that if  $|z - z_0| < \delta$  then

$$\begin{aligned} \left| \frac{f(z) - f(z_0)}{z - z_0} - \sum_{n=1}^{\infty} na_n z_0^{n-1} \right| &\leq |p_N(z)| + |q_N(z)| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$



**Figure 2.8** Points  $z$  and  $z_0$  that satisfy  $|z - z_0| < \delta$

The following corollary to the Differentiation Rule shows that any power series can be integrated term by term on its disc of convergence.

**Corollary Integration Rule for Power Series**

The power series

$$\sum_{n=0}^{\infty} a_n(z - \alpha)^n \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{a_n}{n+1}(z - \alpha)^{n+1}$$

have the same radius of convergence  $R$ .

Furthermore, if  $f(z) = \sum_{n=0}^{\infty} a_n(z - \alpha)^n$ , then the function

$$F(z) = b_0 + \sum_{n=0}^{\infty} \frac{a_n}{n+1}(z - \alpha)^{n+1}, \quad \text{where } b_0 \text{ is any constant,}$$

is a primitive of  $f$  on  $\{z : |z - \alpha| < R\}$ .

**Proof** In order to see how this corollary follows from the Differentiation Rule, we let

$$b_n = \frac{a_{n-1}}{n}, \quad \text{for } n = 1, 2, \dots$$

Then

$$F(z) = \sum_{n=0}^{\infty} b_n(z - \alpha)^n \quad \text{and} \quad f(z) = \sum_{n=1}^{\infty} n b_n(z - \alpha)^{n-1}.$$

It is now clear from the Differentiation Rule that these two power series have the same radius of convergence  $R$ . It is also clear that  $F$  is analytic on  $\{z : |z - \alpha| < R\}$ , and that

$$F'(z) = \sum_{n=1}^{\infty} n b_n(z - \alpha)^{n-1} = f(z), \quad \text{for } |z - \alpha| < R.$$

In other words,  $F$  is a primitive of  $f$  on  $\{z : |z - \alpha| < R\}$ . ■

**Example 2.4**

Use the Integration Rule and the geometric series

$$1 - z + z^2 - z^3 + z^4 - \dots \tag{2.2}$$

to prove that

$$\text{Log}(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \dots, \quad \text{for } |z| < 1.$$

**Solution**

The geometric series (2.2) has radius of convergence 1, so, by the Integration Rule, the power series

$$z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \dots,$$

obtained by integrating (2.2) term by term, has radius of convergence 1.

Let  $f(z) = 1 - z + z^2 - z^3 + z^4 - \dots$ . Then, by the Integration Rule, the function

$$F(z) = b_0 + z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \dots, \quad \text{where } b_0 \in \mathbb{C},$$

is a primitive of  $f$  on  $\{z : |z| < 1\}$ . But  $f(z) = 1/(1+z)$ , for  $|z| < 1$ , so

$$z \mapsto \text{Log}(1+z)$$

is also a primitive of  $f$  on  $\{z : |z| < 1\}$ . Hence

$$\text{Log}(1+z) = b_0 + z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \dots,$$

for some  $b_0$ . Substituting  $z = 0$ , we see that  $b_0 = \text{Log } 1 = 0$ , so

$$\text{Log}(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \dots, \quad \text{for } |z| < 1.$$

We return to this example at the beginning of the next section.

### Exercise 2.5

Use the Differentiation and Integration Rules and the geometric series

$$\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \dots$$

to find the disc of convergence and associated sum function of each of the following power series.

$$(a) \sum_{n=0}^{\infty} (n+1)z^n \quad (b) \sum_{n=0}^{\infty} (n+1)(n+2)z^n \quad (c) \sum_{n=1}^{\infty} \frac{z^n}{n}$$

## Further exercises

### Exercise 2.6

For each of the following power series, determine the radius of convergence and the disc of convergence.

$$(a) \sum_{n=0}^{\infty} (-z)^n \quad (b) \sum_{n=0}^{\infty} (3iz)^n \quad (c) \sum_{n=0}^{\infty} (3i - z)^n \quad (d) \sum_{n=0}^{\infty} (2z - i)^n$$

$$(e) \sum_{n=0}^{\infty} nz^n \quad (f) \sum_{n=1}^{\infty} n!(z+1)^n \quad (g) \sum_{n=1}^{\infty} \left(\frac{z}{n}\right)^n \quad (h) \sum_{n=1}^{\infty} \frac{(z-\pi)^n}{n!}$$

### 3 Taylor's Theorem

After working through this section, you should be able to:

- state Taylor's Theorem
- find the *Taylor series* about a point for an analytic function
- show that the Taylor series about a point for an analytic function  $f$  converges to  $f(z)$  for  $z$  in some suitably chosen disc.

#### 3.1 Taylor series

In Example 2.4 at the end of the previous section you saw that

$$\operatorname{Log}(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots, \quad \text{for } |z| < 1.$$

We say that the function

$$F(z) = \operatorname{Log}(1+z),$$

which has domain  $\mathbb{C} - \{x \in \mathbb{R} : x \leq -1\}$ , is *represented* on the open disc  $D = \{z : |z| < 1\}$  by the power series

$$z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots$$

(see Figure 3.1).

By the Differentiation Rule, any function that can be represented by a power series on an open disc  $D$  in this way must be analytic on  $D$ . But what is more remarkable is the fact that this process can be reversed. Any function that is analytic on an open disc  $D$  can be represented by a power series on  $D$  that is 'Taylor-made' for the purpose.

#### Theorem 3.1 Taylor's Theorem

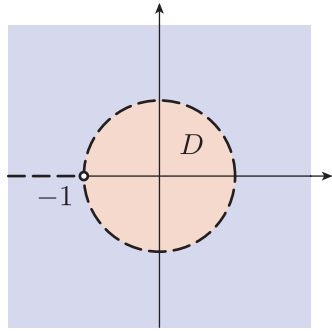
Let  $f$  be a function that is analytic on the open disc  $D = \{z : |z - \alpha| < r\}$ . Then

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(\alpha)}{n!} (z - \alpha)^n, \quad \text{for } z \in D.$$

Moreover, this representation of  $f$  is unique, in the sense that if

$$f(z) = \sum_{n=0}^{\infty} a_n (z - \alpha)^n, \quad \text{for } z \in D,$$

then  $a_n = f^{(n)}(\alpha)/n!$ , for  $n = 0, 1, 2, \dots$ .



**Figure 3.1** The open disc  $D = \{z : |z| < 1\}$  inside  $\mathbb{C} - \{x \in \mathbb{R} : x \leq -1\}$

Taylor's Theorem will be proved in Subsection 3.3.

### Remarks

1. The term  $f^{(n)}(\alpha)/n!$  makes sense for  $n = 0$  because, by convention, we take  $0! = 1$  and  $f^{(0)}(z) = f(z)$ .
2. The uniqueness of the power series representation of a function about a given point is an important result, which we will use often.
3. Taylor's Theorem asserts the *equality* of the value of the given function  $f$  and of the power series for each value of  $z$  in  $D$ . This is in contrast to Section 2 where, in writing

$$f(z) = \sum_{n=0}^{\infty} a_n(z - \alpha)^n,$$

we were using the power series to *define* the function  $f$ .

4. Taylor's Theorem shows that a function  $f$  that is analytic at  $\alpha$  can be represented by a power series about  $\alpha$ , whose coefficients are of the form  $f^{(n)}(\alpha)/n!$ . Of course, this can be the case only if  $f$  has derivatives of all orders at  $\alpha$ , a result which was stated (but not proved) as part of Cauchy's  $n$ th Derivative Formula (Theorem 3.2 of Unit B2). We will obtain a proof of Cauchy's  $n$ th Derivative Formula from the proof of Taylor's Theorem.
5. Taylor's Theorem is an extremely useful result which enables us to use power series to investigate the properties of analytic functions. You will see several examples of this later in the unit, and many more throughout the module.

### Definitions

Let  $f$  be a function with derivatives  $f^{(1)}(\alpha), f^{(2)}(\alpha), f^{(3)}(\alpha), \dots$  at the point  $\alpha$ . Then the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(\alpha)}{n!} (z - \alpha)^n$$

is called the **Taylor series about  $\alpha$  for  $f$** . The coefficient  $f^{(n)}(\alpha)/n!$  is known as the  **$n$ th Taylor coefficient of  $f$  at  $\alpha$** .

Some texts refer to the Taylor series about 0 for  $f$  as the *Maclaurin series* for  $f$ , named after the Scottish mathematician Colin Maclaurin (1698–1746) who studied such series.

**Remark**

Suppose that  $f$  is analytic on the open disc  $D = \{z : |z - \alpha| < r\}$ . By Cauchy's  $n$ th Derivative Formula, we can write the  $n$ th Taylor coefficient of  $f$  at  $\alpha$  in the alternative form

$$\frac{f^{(n)}(\alpha)}{n!} = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - \alpha)^{n+1}} dz,$$

where  $C$  is a circle centred at  $\alpha$  with radius less than  $r$ .

Sometimes it is possible to find a Taylor series for a function  $f$  directly from the definition.

**Example 3.1**

Find the Taylor series about 0 for the function  $f(z) = e^z$ .

**Solution**

The Taylor series is  $\sum_{n=0}^{\infty} a_n z^n$ , where  $a_n = f^{(n)}(0)/n!$ .

Since the exponential function is its own derivative, all the higher derivatives  $f^{(n)}(z)$  are equal to  $e^z$ , so

$$f^{(n)}(0) = e^0 = 1.$$

The Taylor series about 0 for the exponential function is therefore

$$1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots.$$

**Remarks**

1. In this example we have given the first four terms of the Taylor series followed by three dots  $\cdots$ , to indicate that the pattern continues in a similar way. Generally, there is no hard-and-fast rule for how many terms to include when writing down a Taylor series; unless an exercise tells you how many terms to include, you should just put enough to make it clear how the sequence of Taylor coefficients carries on.
2. You will notice also that we represented the coefficients using factorial notation, rather than 'simplifying' each coefficient to give

$$1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots.$$

The reason for *not* 'simplifying' the coefficients is because it is far less clear from this alternative representation how the sequence of Taylor coefficients continues.

Since the function  $f(z) = e^z$  in Example 3.1 is entire, it must be analytic on every open disc centred at 0. So the series in the example must converge to  $e^z$  for each  $z \in \mathbb{C}$ .

More generally we have the following corollary to Taylor's Theorem.

### Corollary

Let  $f$  be an entire function. Then for any point  $\alpha$ , the Taylor series about  $\alpha$  for  $f$  converges to  $f(z)$  for each  $z \in \mathbb{C}$ .

### Example 3.2

Find the Taylor series about 0 for the function  $f(z) = \cos z$ . Explain why the series converges to  $\cos z$  for each  $z \in \mathbb{C}$ .

### Solution

Since the function  $f(z) = \cos z$  is entire, it follows from the corollary that its Taylor series about 0 must converge to  $\cos z$  for each  $z \in \mathbb{C}$ . The Taylor series is found by calculating the higher derivatives of  $f$  at 0:

$$\begin{aligned} f(z) &= \cos z, & \text{so } f(0) &= 1, \\ f^{(1)}(z) &= -\sin z, & \text{so } f^{(1)}(0) &= 0, \\ f^{(2)}(z) &= -\cos z, & \text{so } f^{(2)}(0) &= -1, \\ f^{(3)}(z) &= \sin z, & \text{so } f^{(3)}(0) &= 0, \\ f^{(4)}(z) &= \cos z, & \text{so } f^{(4)}(0) &= 1. \end{aligned}$$

Since every fourth differentiation brings us back to  $f(z)$ , the pattern above repeats itself. The Taylor series about 0 for the function  $f$  is therefore

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots, \quad \text{for } z \in \mathbb{C}.$$

Notice that all the Taylor coefficients of  $\cos$  at 0 are real numbers (as were the Taylor coefficients of  $\exp$  at 0 in Example 3.1). This is because  $\cos$  is a real function when restricted to the real line, so the higher derivatives of  $\cos$  at 0 are all real numbers. For similar reasons, you should find that all the Taylor coefficients for the series that you obtain in the next exercise are real too.

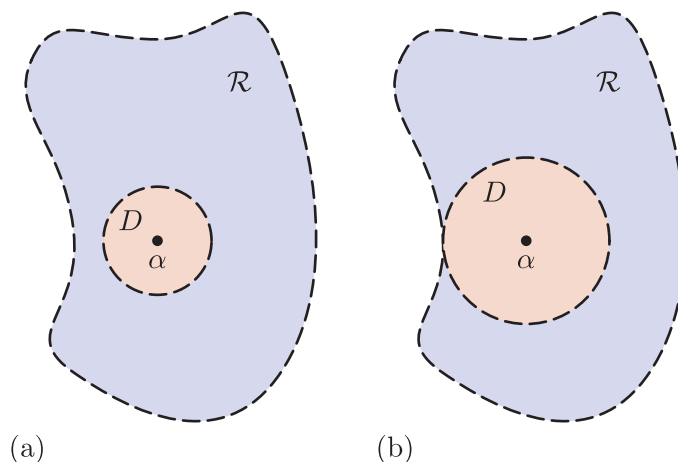
### Exercise 3.1

For each of the following functions  $f$ , find the Taylor series about 0 for  $f$ , and explain why it converges to  $f(z)$  for each  $z \in \mathbb{C}$ .

- (a)  $f(z) = \sin z$       (b)  $f(z) = \cosh z$       (c)  $f(z) = \sinh z$

If  $f$  is not entire, then it may not be possible to find a Taylor series that converges to  $f(z)$  for each  $z$  in the domain of  $f$ . In such cases, it often

helps to identify a region  $\mathcal{R}$  on which  $f$  is analytic. We can then pick a point  $\alpha$  in  $\mathcal{R}$  and try to find the Taylor series about  $\alpha$  for  $f$ . By Taylor's Theorem, this series converges to  $f(z)$  for each  $z$  in any open disc  $D$  centred at  $\alpha$  that lies within the region  $\mathcal{R}$  (see Figure 3.2(a)). We often choose  $D$  to be as large as possible (see Figure 3.2(b)).



**Figure 3.2** (a) An open disc  $D$  centred at  $\alpha$  (b) The largest open disc  $D$  centred at  $\alpha$  that is contained in  $\mathcal{R}$

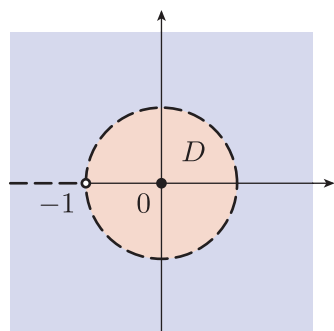
As an example, consider the Taylor series about 0 for the function

$$f(z) = \text{Log}(1 + z),$$

which is not entire.

In Example 2.4 we used the Integration Rule to obtain a power series about 0 representing  $f$ . Taylor's Theorem tells us that  $f$  is represented by only one power series about 0 (the representation is unique), so the power series we obtained in Example 2.4 must be the Taylor series about 0 for  $f$ .

Let us find the same Taylor series again, this time by calculating the Taylor coefficients and applying Taylor's Theorem.



**Figure 3.3** The open disc  $D = \{z : |z| < 1\}$  inside  $\mathbb{C} - \{x \in \mathbb{R} : x \leq -1\}$

### Example 3.3

Calculate the  $n$ th Taylor coefficient of the function  $f(z) = \text{Log}(1 + z)$  at 0, and hence find the Taylor series about 0 for  $f$ .

Show that the series converges to  $\text{Log}(1 + z)$  for  $|z| < 1$ .

### Solution

The function  $f$  is analytic on the region  $\mathbb{C} - \{x \in \mathbb{R} : x \leq -1\}$ . The largest open disc centred at 0 that will fit in this region is  $D = \{z : |z| < 1\}$ , as illustrated in Figure 3.3. So, by Taylor's Theorem, the Taylor series about 0 for  $f$  converges to  $f(z)$ , for  $|z| < 1$ .

The Taylor series is found by calculating the higher derivatives of  $f$  at 0:

$$\begin{aligned} f(z) &= \text{Log}(1+z), & \text{so } f(0) &= 0, \\ f^{(1)}(z) &= (1+z)^{-1}, & \text{so } f^{(1)}(0) &= 1, \\ f^{(2)}(z) &= -(1+z)^{-2}, & \text{so } f^{(2)}(0) &= -1, \\ f^{(3)}(z) &= 2(1+z)^{-3}, & \text{so } f^{(3)}(0) &= 2, \\ f^{(4)}(z) &= -3 \times 2(1+z)^{-4}, & \text{so } f^{(4)}(0) &= -3!. \end{aligned}$$

Each differentiation multiplies the coefficient of  $(1+z)$  by the current power of  $(1+z)$  and reduces the power of  $(1+z)$  by 1. Therefore, in general,

$$f^{(n)}(z) = \frac{(-1)^{n+1}(n-1)!}{(1+z)^n},$$

so

$$\frac{f^{(n)}(0)}{n!} = \frac{(-1)^{n+1}}{n}, \quad \text{for } n = 1, 2, \dots,$$

which could be established formally by the Principle of Mathematical Induction. Thus the Taylor series about 0 for  $f$  is

$$\text{Log}(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \dots, \quad \text{for } |z| < 1.$$

This series is indeed the same as the one that we found in Example 2.4.

Sometimes it is clearer to indicate how a series continues by giving an expression for the general term. For example, in Example 3.3 we could have written

$$\text{Log}(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \dots + \frac{(-1)^{n+1}z^n}{n} + \dots,$$

or, equivalently,

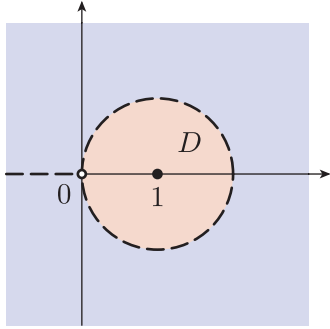
$$\text{Log}(1+z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}z^n}{n}.$$

### Exercise 3.2

Find the Taylor series about 0 for the function  $f(z) = (1+z)^{-3}$ , giving an expression for the general term.

Show that the series converges to  $(1+z)^{-3}$  for  $|z| < 1$ .

So far, all of the Taylor series we have found have been about the point 0, but Taylor series about other points can be obtained in a similar way. The following example looks at the function  $f(z) = \text{Log } z$ . This function is not analytic at 0, but it is analytic at 1, so it has a Taylor series about 1.



**Figure 3.4** The open disc  $D = \{z : |z - 1| < 1\}$  inside  $\mathbb{C} - \{x \in \mathbb{R} : x \leq 0\}$

### Example 3.4

Find the Taylor series about 1 for the function  $f(z) = \text{Log } z$ .

Show that the series converges to  $\text{Log } z$  for  $|z - 1| < 1$ .

### Solution

The function  $f$  is analytic on the region  $\mathbb{C} - \{x \in \mathbb{R} : x \leq 0\}$ . The largest open disc centred at 1 that will fit in this region is

$D = \{z : |z - 1| < 1\}$ , as illustrated in Figure 3.4. So, by Taylor's Theorem, the Taylor series about 1 for  $f$  converges to  $f(z)$  for  $|z - 1| < 1$ .

The Taylor series is found by calculating the higher derivatives of  $f$  at 1:

$$\begin{aligned} f(z) &= \text{Log } z, & \text{so } f(1) &= \text{Log } 1 = 0, \\ f^{(1)}(z) &= z^{-1}, & \text{so } f^{(1)}(1) &= 1, \\ f^{(2)}(z) &= -z^{-2}, & \text{so } f^{(2)}(1) &= -1, \\ f^{(3)}(z) &= 2z^{-3}, & \text{so } f^{(3)}(1) &= 2, \\ f^{(4)}(z) &= -3 \times 2z^{-4}, & \text{so } f^{(4)}(1) &= -3!. \end{aligned}$$

In general,

$$f^{(n)}(z) = \frac{(-1)^{n+1}(n-1)!}{z^n},$$

so

$$\frac{f^{(n)}(1)}{n!} = \frac{(-1)^{n+1}}{n}, \quad \text{for } n = 1, 2, \dots$$

Thus the Taylor series about 1 for  $f$  is

$$\text{Log } z = (z - 1) - \frac{(z - 1)^2}{2} + \frac{(z - 1)^3}{3} - \frac{(z - 1)^4}{4} + \dots,$$

for  $|z - 1| < 1$ .

The terms of this power series are each multiples of  $(z - 1)^n$  (not  $z^n$ ) since we are asked for a Taylor series about 1 (not about 0).

You may notice that this series is similar to the series

$$\text{Log}(1 + w) = w - \frac{w^2}{2} + \frac{w^3}{3} - \frac{w^4}{4} + \dots, \quad \text{for } |w| < 1, \quad (3.1)$$

found in Example 3.3 (expressed here with  $w$  in place of  $z$ , for reasons to emerge shortly). In fact, sometimes the easiest way to obtain a Taylor series about a point  $\alpha$  other than 0 is to apply a substitution that allows you to obtain the Taylor series about  $\alpha$  from a known Taylor series about 0. For example, to obtain the Taylor series about 1 for  $f(z) = \text{Log } z$ , we first write

$$f(z) = \text{Log } z = \text{Log}(1 + (z - 1)).$$

Now let  $w = z - 1$ , in which case  $|w| < 1$  if and only if  $|z - 1| < 1$ . Then,

using equation (3.1), we see that

$$\operatorname{Log} z = \operatorname{Log}(1 + w) = w - \frac{w^2}{2} + \frac{w^3}{3} - \frac{w^4}{4} + \cdots, \quad \text{for } |w| < 1,$$

and on substituting  $w = z - 1$ , we obtain the Taylor series about 1 found in Example 3.4.

This method will be explained in more detail in Subsection 4.1 when we study the Substitution Rule for power series.

In general, to calculate a Taylor series about a point  $\alpha$  other than 0, you can either calculate the Taylor coefficients in the manner of Example 3.4, or apply a substitution to replace  $\alpha$  by 0; the suitability of each method depends on the function in question (often both work well).

### Exercise 3.3

Find the Taylor series about  $i$  for the function  $f(z) = 1/z$ , giving an expression for the general term.

We end this subsection by briefly discussing the useful concepts of *even functions* and *odd functions*, and interpreting the significance of these concepts in terms of Taylor series.

### Definitions

Let  $A$  be a set for which  $z \in A$  if and only if  $-z \in A$ .

A function  $f: A \rightarrow \mathbb{C}$  is an **even function** if

$$f(-z) = f(z), \quad \text{for } z \in A,$$

and  $f$  is an **odd function** if

$$f(-z) = -f(z), \quad \text{for } z \in A.$$

For example,  $\cos$  is an even function, and  $\sin$  is an odd function.

The Taylor series about 0 for an even or odd function takes a special form.

### Theorem 3.2

Let  $f$  be a function that is analytic at 0 with Taylor series about 0 given by

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

(a) If  $f$  is an even function, then  $a_n = 0$  for  $n$  odd.

(b) If  $f$  is an odd function, then  $a_n = 0$  for  $n$  even.

Thus if  $f$  is even, then its Taylor series about 0 has only even powers, and if  $f$  is odd, then its Taylor series about 0 has only odd powers.

**Proof** Choose an open disc  $D = \{z : |z| < r\}$  on which the Taylor series about 0 for  $f$  converges.

(a) Suppose that  $f$  is an even function. If  $z \in D$ , then

$$f(z) = f(-z) = \sum_{n=0}^{\infty} a_n(-z)^n = \sum_{n=0}^{\infty} (-1)^n a_n z^n.$$

By the uniqueness of Taylor series (Taylor's Theorem, Theorem 3.1),

$$(-1)^n a_n = a_n, \quad \text{for } n = 0, 1, 2, \dots,$$

so  $a_n = 0$ , for  $n$  odd.

(b) Suppose that  $f$  is an odd function. If  $z \in D$ , then

$$f(z) = -f(-z) = -\sum_{n=0}^{\infty} a_n(-z)^n = -\sum_{n=0}^{\infty} (-1)^n a_n z^n.$$

By the uniqueness of Taylor series,

$$(-1)^n a_n = -a_n, \quad \text{for } n = 0, 1, 2, \dots,$$

so  $a_n = 0$ , for  $n$  even. ■

There is a converse result to Theorem 3.2(a), which says that if  $f$  is an analytic function with domain an open disc  $\{z : |z| < r\}$ , and

$$f(z) = a_0 + a_2 z^2 + a_4 z^4 + a_6 z^6 + \dots, \quad \text{for } |z| < r$$

(no odd powers), then  $f$  is an even function. This can be established by simply substituting  $-z$  for  $z$  in the series above. A similar converse result can be stated for Theorem 3.2(b).

Later on we make use of the result of the following exercise.

### Exercise 3.4

Let  $f: A \rightarrow \mathbb{C}$  be a one-to-one function. Prove that if  $f$  is an odd function, then the inverse function  $f^{-1}: f(A) \rightarrow A$  is also an odd function.

## 3.2 Basic Taylor series

In principle we could use Taylor's Theorem to calculate Taylor series for any given function. However, in practice it is not always easy to find the higher derivatives of the function concerned. In the next section we illustrate how it is often easier to calculate a Taylor series by applying the rules for series (discussed in Sections 1 and 2) to a list of basic functions whose Taylor series are already known. Some frequently occurring Taylor series (about 0) that can be used for this purpose are listed below.

**Basic Taylor series**

$$(1 - z)^{-1} = 1 + z + z^2 + z^3 + \cdots, \quad \text{for } |z| < 1$$

$$\exp z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots, \quad \text{for } z \in \mathbb{C}$$

$$\operatorname{Log}(1 + z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots, \quad \text{for } |z| < 1$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots, \quad \text{for } z \in \mathbb{C}$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots, \quad \text{for } z \in \mathbb{C}$$

$$\sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + \cdots, \quad \text{for } z \in \mathbb{C}$$

$$\cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \cdots, \quad \text{for } z \in \mathbb{C}$$

In addition to these basic Taylor series, we frequently use Taylor series about 0 for functions of the form  $f(z) = (1 + z)^\alpha$ , where  $\alpha$  is a complex number, which are all analytic on  $\mathbb{C} - \{x : x \leq -1\}$ . Such Taylor series are known as **binomial series**.

You have already investigated one binomial series in Exercise 3.2, where you found that

$$(1 + z)^{-3} = 1 - \frac{3 \times 2}{2}z + \frac{4 \times 3}{2}z^2 - \frac{5 \times 4}{2}z^3 + \cdots, \quad \text{for } |z| < 1.$$

More generally, we have the following result, which is a generalisation of the Binomial Theorem (Theorem 1.2 of Unit A1). (The Binomial Theorem is the special case in which  $\alpha$  is a positive integer.)

**Theorem 3.3 Binomial Series**

Let  $\alpha \in \mathbb{C}$ . The binomial series about 0 for the function  $f(z) = (1 + z)^\alpha$  is

$$(1 + z)^\alpha = 1 + \binom{\alpha}{1}z + \binom{\alpha}{2}z^2 + \binom{\alpha}{3}z^3 + \cdots, \quad \text{for } |z| < 1,$$

where

$$\binom{\alpha}{n} = \frac{\alpha(\alpha - 1)(\alpha - 2) \cdots (\alpha - (n - 1))}{n!}.$$

If  $\alpha$  is a positive integer or zero, then the binomial series reduces to a polynomial; otherwise, the series is a power series whose radius of convergence is 1.

The coefficients  $\binom{\alpha}{n}$  are called the **binomial coefficients** of the binomial series.

**Proof** We have

$$\begin{aligned}
 f(z) &= (1+z)^\alpha, & \text{so } f(0) &= 1, \\
 f^{(1)}(z) &= \alpha(1+z)^{\alpha-1}, & \text{so } f^{(1)}(0) &= \alpha, \\
 f^{(2)}(z) &= \alpha(\alpha-1)(1+z)^{\alpha-2}, & \text{so } f^{(2)}(0) &= \alpha(\alpha-1), \\
 f^{(3)}(z) &= \alpha(\alpha-1)(\alpha-2)(1+z)^{\alpha-3}, & \text{so } f^{(3)}(0) &= \alpha(\alpha-1)(\alpha-2).
 \end{aligned}$$

In general, the  $n$ th Taylor coefficient of  $f$  at 0 is

$$\frac{f^{(n)}(0)}{n!} = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-(n-1))}{n!} = \binom{\alpha}{n}.$$

The function  $f(z) = (1+z)^\alpha = \exp(\alpha \operatorname{Log}(1+z))$  is analytic on the open disc  $D = \{z : |z| < 1\}$ , so, by Taylor's Theorem, the binomial series converges to  $(1+z)^\alpha$  for  $|z| < 1$ . So the radius of convergence is at least 1.

If  $\alpha$  is a positive integer or 0, then all but finitely many of the binomial coefficients are 0, and  $f$  is a polynomial function. Otherwise, the binomial coefficients are non-zero, and then the Radius of Convergence Formula (Theorem 2.2) tells us that the radius of convergence of the binomial series is 1, because

$$\begin{aligned}
 \left| \binom{\alpha}{n} \right| / \left| \binom{\alpha}{n+1} \right| &= \left| \frac{\alpha(\alpha-1)\cdots(\alpha-(n-1))(n+1)!}{\alpha(\alpha-1)\cdots(\alpha-n)n!} \right| \\
 &= \left| \frac{n+1}{\alpha-n} \right| \\
 &= \left| \frac{1+1/n}{\alpha/n-1} \right| \rightarrow 1 \text{ as } n \rightarrow \infty,
 \end{aligned}$$

as required. ■

The next example demonstrates how to use binomial series.

### Example 3.5

Find the Taylor series about 0 for the function  $f(z) = (1+z)^{-2}$ .

#### Solution

We have

$$\binom{-2}{n} = \frac{(-2) \times (-3) \times (-4) \times \cdots \times (-(n+1))}{n!} = (-1)^n (n+1).$$

Hence, by Theorem 3.3 on binomial series,

$$(1+z)^{-2} = 1 - 2z + 3z^2 - 4z^3 + \cdots, \quad \text{for } |z| < 1.$$

In Example 3.5 the exponent  $\alpha$  was the integer  $-2$ , but binomial series can be used for any value of  $\alpha$ , real or complex.

**Exercise 3.5**

Find the Taylor series about 0 for the function  $f(z) = (1+z)^i$ . Give the first four terms of the series, with each Taylor coefficient in Cartesian form.

**3.3 Proof of Taylor's Theorem**

To end this section we prove Taylor's Theorem and then deduce Cauchy's  $n$ th Derivative Formula, which was introduced without proof in Unit B2. The details of these proofs are challenging, so you may wish to omit them on a first reading.

**Theorem 3.1 Taylor's Theorem**

Let  $f$  be a function that is analytic on the open disc  $D = \{z : |z - \alpha| < r\}$ . Then

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(\alpha)}{n!} (z - \alpha)^n, \quad \text{for } z \in D.$$

Moreover, this representation of  $f$  is unique, in the sense that if

$$f(z) = \sum_{n=0}^{\infty} a_n (z - \alpha)^n, \quad \text{for } z \in D,$$

then  $a_n = f^{(n)}(\alpha)/n!$ , for  $n = 0, 1, 2, \dots$

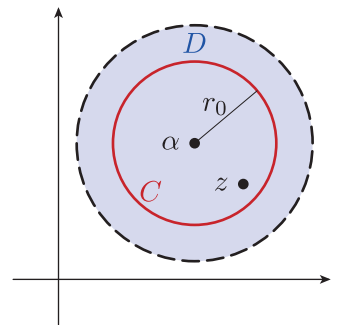
**Proof** There are six steps in the proof. We first represent  $f$  as an integral, then in step 2 we approximate the integrand by a polynomial in  $z - \alpha$  of degree  $n$ . In step 3 we integrate the polynomial term by term, and then in step 4 we obtain a power series representation of  $f$  by letting  $n \rightarrow \infty$ . In step 5 we complete the proof by differentiating the series  $n$  times; this shows that the coefficients of the series are  $f^{(n)}(\alpha)/n!$ . We use this result in step 6 to show that the coefficients are unique.

1. Let  $z$  be an arbitrary point in  $D$ . By choosing  $r_0$  such that  $|z - \alpha| < r_0 < r$ , we can ensure that the circle  $C$  with centre  $\alpha$  and radius  $r_0$  lies in  $D$ , and encloses the point  $z$  (see Figure 3.5). By applying Cauchy's Integral Formula (Theorem 2.1 of Unit B2) to the function  $f$ , we can express  $f(z)$  in terms of the values of  $f$  on the circle  $C$ :

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w - z} dw.$$

2. Next, we note that, for  $w \in C$ ,

$$\frac{1}{w - z} = \frac{1}{(w - \alpha) - (z - \alpha)} = \frac{1}{w - \alpha} \left( 1 - \frac{z - \alpha}{w - \alpha} \right)^{-1}. \quad (3.2)$$



**Figure 3.5** Circle  $C$  inside  $D$

But for any  $\lambda \in \mathbb{C}$ ,  $n \in \mathbb{N}$ , we have

$$(1 - \lambda)^{-1} = 1 + \lambda + \lambda^2 + \cdots + \lambda^{n-1} + \frac{\lambda^n}{1 - \lambda},$$

as you can check by multiplying both sides of the equation by  $1 - \lambda$ .

Replacing  $\lambda$  by  $(z - \alpha)/(w - \alpha)$ , we see that

$$\left(1 - \frac{z - \alpha}{w - \alpha}\right)^{-1}$$

is equal to

$$1 + \frac{z - \alpha}{w - \alpha} + \cdots + \frac{(z - \alpha)^{n-1}}{(w - \alpha)^{n-1}} + \frac{((z - \alpha)/(w - \alpha))^n}{1 - (z - \alpha)/(w - \alpha)}.$$

So, from equation (3.2), we have

$$\frac{1}{w - z} = \frac{1}{w - \alpha} + \frac{z - \alpha}{(w - \alpha)^2} + \cdots + \frac{(z - \alpha)^{n-1}}{(w - \alpha)^n} + \frac{((z - \alpha)/(w - \alpha))^n}{(w - \alpha) - (z - \alpha)}.$$

3. Substituting the expression above for  $1/(w - z)$  into the integral for  $f(z)$  from step 1, we obtain

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_C \frac{f(w)}{w - \alpha} dw + \frac{z - \alpha}{2\pi i} \int_C \frac{f(w)}{(w - \alpha)^2} dw + \cdots \\ &\quad + \frac{(z - \alpha)^{n-1}}{2\pi i} \int_C \frac{f(w)}{(w - \alpha)^n} dw \\ &\quad + \frac{(z - \alpha)^n}{2\pi i} \int_C \frac{f(w)}{(w - \alpha)^n(w - z)} dw. \end{aligned}$$

We now write this in the form

$$f(z) = b_0 + b_1(z - \alpha) + b_2(z - \alpha)^2 + \cdots + b_{n-1}(z - \alpha)^{n-1} + I_n(z),$$

where

$$b_m = \frac{1}{2\pi i} \int_C \frac{f(w)}{(w - \alpha)^{m+1}} dw, \quad \text{for } m = 0, 1, \dots, \quad (3.3)$$

and

$$I_n(z) = \frac{(z - \alpha)^n}{2\pi i} \int_C \frac{f(w)}{(w - \alpha)^n(w - z)} dw.$$

4. Next, we use the Estimation Theorem (Theorem 4.1 of Unit B1) to show that  $I_n(z) \rightarrow 0$  as  $n \rightarrow \infty$  (remember that  $z$  was fixed in step 1). Since  $f$  is analytic on  $D$ , we see that  $f$  is continuous on the compact set  $C$ , and thus  $f$  is bounded on  $C$  by the Boundedness Theorem (Theorem 5.3 of Unit A3). That is, there is a number  $M$  such that  $|f(w)| \leq M$  for  $w \in C$ . Also, for  $w \in C$ , we see on writing

$$w - z = (w - \alpha) - (z - \alpha)$$

and applying the backwards form of the Triangle Inequality that

$$|w - z| \geq |w - \alpha| - |z - \alpha| = r_0 - |z - \alpha|.$$

Hence it follows from the Estimation Theorem that

$$\begin{aligned} |I_n(z)| &\leq \frac{|z - \alpha|^n}{2\pi} \times \frac{M}{r_0^n(r_0 - |z - \alpha|)} \times 2\pi r_0 \\ &= \frac{Mr_0}{r_0 - |z - \alpha|} \left| \frac{z - \alpha}{r_0} \right|^n. \end{aligned}$$

But  $|z - \alpha| < r_0$ , so the right-hand side tends to 0 as  $n \rightarrow \infty$ .

Hence  $I_n(z) \rightarrow 0$  as  $n \rightarrow \infty$  (by the Squeeze Rule, Theorem 1.1 of Unit A3). Therefore

$$f(z) = \sum_{n=0}^{\infty} b_n(z - \alpha)^n, \quad \text{for } z \in D,$$

where  $b_0, b_1, b_2, \dots$  are given by equation (3.3).

5. By the Differentiation Rule applied to the Taylor series about  $\alpha$  for  $f$ , we see that  $f$  must have derivatives of all orders at  $\alpha$ , and the coefficients  $b_n$  must be equal to  $f^{(n)}(\alpha)/n!$ . Indeed, for  $z \in D$ ,

$$\begin{aligned} f(z) &= b_0 + b_1(z - \alpha) + b_2(z - \alpha)^2 + \cdots, \\ f^{(1)}(z) &= 1b_1 + 2b_2(z - \alpha) + 3b_3(z - \alpha)^2 + \cdots, \\ f^{(2)}(z) &= 2 \cdot 1b_2 + 3 \cdot 2b_3(z - \alpha) + 4 \cdot 3b_4(z - \alpha)^2 + \cdots, \\ f^{(3)}(z) &= 3 \cdot 2 \cdot 1b_3 + 4 \cdot 3 \cdot 2b_4(z - \alpha) + 5 \cdot 4 \cdot 3b_5(z - \alpha)^2 + \cdots, \\ &\vdots \end{aligned}$$

It follows that  $f^{(1)}(\alpha) = 1!b_1$ ,  $f^{(2)}(\alpha) = 2!b_2$ ,  $f^{(3)}(\alpha) = 3!b_3$ , and, in general,  $f^{(n)}(\alpha) = n!b_n$ , for  $n = 1, 2, \dots$ , as required.

(This formula for  $f^{(n)}(\alpha)$  could be established formally by the Principle of Mathematical Induction.)

6. Finally, notice that the same differentiation argument shows that the representation is unique. For if

$$f(z) = \sum_{n=0}^{\infty} a_n(z - \alpha)^n, \quad \text{for } z \in D,$$

then  $a_n = f^{(n)}(\alpha)/n! = b_n$ , for  $n = 1, 2, \dots$ . ■

With remarkably little further effort we can prove Cauchy's  $n$ th Derivative Formula (Theorem 3.2 of Unit B2).

### Cauchy's $n$ th Derivative Formula

Let  $\mathcal{R}$  be a simply connected region, let  $\Gamma$  be a simple-closed contour in  $\mathcal{R}$ , and let  $f$  be a function that is analytic on  $\mathcal{R}$ . Then, for any point  $\alpha$  inside  $\Gamma$ ,  $f$  is  $n$ -times differentiable at  $\alpha$  and

$$f^{(n)}(\alpha) = \frac{n!}{2\pi i} \int_{\Gamma} \frac{f(z)}{(z - \alpha)^{n+1}} dz, \quad \text{for } n = 1, 2, \dots$$

**Proof** Let  $D = \{z : |z - \alpha| < r\}$  be an open disc inside  $\Gamma$  centred at  $\alpha$ . Since  $f$  is analytic on  $D$ , we can proceed to use the same arguments as in the proof of Taylor's Theorem. In particular, we can choose a circle  $C = \{w : |w - \alpha| = r_0\}$  in  $D$  centred at  $\alpha$ , and show that the  $m$ th Taylor coefficient of  $f$  at  $\alpha$  is

$$b_m = \frac{1}{2\pi i} \int_C \frac{f(w)}{(w - \alpha)^{m+1}} dw, \quad \text{for } m = 0, 1, 2, \dots$$

But from step 5 of the proof of Taylor's Theorem, we know that  $f$  is  $n$ -times differentiable at  $\alpha$  and  $f^{(n)}(\alpha) = n! b_n$ , so

$$f^{(n)}(\alpha) = \frac{n!}{2\pi i} \int_C \frac{f(w)}{(w - \alpha)^{n+1}} dw, \quad \text{for } n = 1, 2, \dots$$

Since  $\mathcal{R}$  is simply connected, and  $C$  lies inside  $\Gamma$ , we can deduce Cauchy's  $n$ th Derivative Formula by applying the Shrinking Contour Theorem (Theorem 1.4 of Unit B2). ■

## Further exercises

### Exercise 3.6

Find the Taylor series about the given point for each of the following functions. In each case give the general term of the series.

- (a)  $f(z) = \sinh 2z$ , about 0
- (b)  $f(z) = z \sin z$ , about 0
- (c)  $f(z) = e^{iz}$ , about  $\pi/4$

### Polynomial approximation

For many practical applications it is useful to be able to approximate complex functions by polynomials, because polynomials are easy to manipulate and evaluate. Taylor series provide a means for finding such approximations.

For example, consider the function

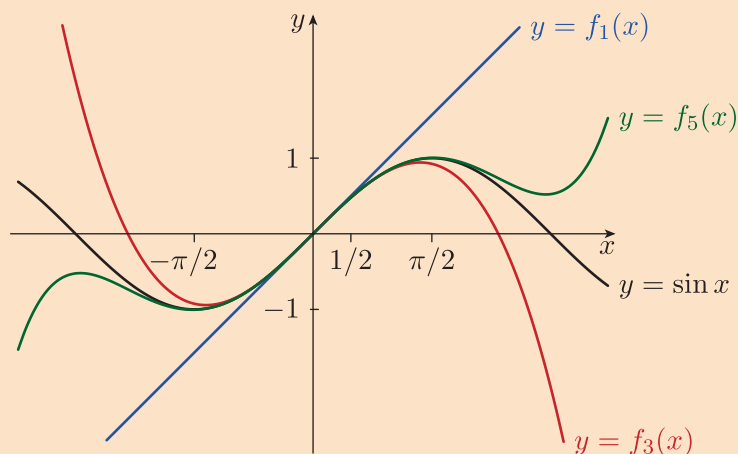
$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots$$

We can try approximating  $\sin$  by the polynomial functions

$$f_1(z) = z, \quad f_3(z) = z - \frac{z^3}{3!}, \quad f_5(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!}, \quad \dots$$

These functions are called the **Taylor polynomials** about 0 for  $\sin$ . Figure 3.6 indicates that, at least for *real* values of  $z$ , the functions  $f_n$

provide reasonable approximations to  $\sin$ , with growing accuracy as  $n$  increases and as  $z$  approaches 0.



**Figure 3.6** Graphs of  $y = \sin x$  and  $y = f_n(x)$ , for  $n = 1, 2, 3$

We can get a better idea of the accuracy of approximating a function by its Taylor polynomials through looking more closely at the proof of Taylor's Theorem. Using a special case of step 4 of that proof, we can show that if  $f$  is an entire function such as  $\sin$ , and  $|f(z)| \leq M$  for  $|z| = 1$ , then we can write

$$f(z) = a_0 + a_1z + a_2z^2 + \cdots + a_nz^n + I_n(z), \quad \text{for } |z| < 1,$$

where

$$|I_n(z)| \leq \frac{M|z|^{n+1}}{1 - |z|}.$$

In particular, if  $f(z) = \sin z$ , then it can be proven that

$$|f(z)| \leq \sinh 1 < 2, \quad \text{for } |z| = 1.$$

Hence, using the function  $f_5$  defined earlier, we see that

$$|\sin z - f_5(z)| = |I_6(z)| \leq \frac{2|z|^6}{1 - |z|}, \quad \text{for } |z| < 1.$$

So if  $|z| < 1/2$ , then

$$|\sin z - f_5(z)| < \frac{2(1/2)^6}{1 - 1/2} = \frac{1}{16},$$

which agrees with Figure 3.6, because the curves  $y = \sin x$  and  $y = f_5(x)$  are almost indistinguishable for  $|x| < 1/2$ .

## 4 Manipulating Taylor series

After working through this section, you should be able to:

- use the Combination Rules for power series
- use the Product Rule for power series
- use the Composition Rule for power series
- use the Differentiation and Integration Rules for power series to determine Taylor series of functions.

### 4.1 Finding Taylor series

In the previous section we used the formula for Taylor coefficients to find a list of basic Taylor series. We were able to do this because it was relatively easy to calculate the higher derivatives of the functions involved.

Unfortunately, for many functions calculating higher derivatives can be messy, which makes the formula difficult to apply.

In this subsection we illustrate how to find the Taylor series for many functions by applying the rules for manipulating power series to the basic Taylor series found in the previous section. Several of the rules that are needed for this purpose were introduced earlier in the unit, where we showed that we can add series, take multiples of series, and differentiate and integrate power series as if they were polynomials. Here this similarity with polynomials is further reinforced when we introduce two more rules which show that power series can be multiplied and composed as if they were polynomials.

To get us started, let us first revisit the Combination Rules for series (Theorem 1.4). For each value of  $z$ , a power series is just a series, so we can obtain a version of the Combination Rules for power series.

#### Theorem 4.1 Combination Rules for Power Series

Let

$$f(z) = \sum_{n=0}^{\infty} a_n(z - \alpha)^n, \quad \text{for } |z - \alpha| < R,$$

$$g(z) = \sum_{n=0}^{\infty} b_n(z - \alpha)^n, \quad \text{for } |z - \alpha| < R'.$$

(a) **Sum Rule** If  $r = \min\{R, R'\}$ , then

$$(f + g)(z) = \sum_{n=0}^{\infty} (a_n + b_n)(z - \alpha)^n, \quad \text{for } |z - \alpha| < r.$$

(b) **Multiple Rule** If  $\lambda \in \mathbb{C}$ , then

$$(\lambda f)(z) = \sum_{n=0}^{\infty} \lambda a_n(z - \alpha)^n, \quad \text{for } |z - \alpha| < R.$$

These rules follow immediately from the earlier Combination Rules for series; the bounds on  $|z - \alpha|$  are there to ensure that the series converge. For the Sum Rule, we need  $|z - \alpha|$  to be smaller than the minimum of  $R$  and  $R'$  to ensure that the series for both  $f$  and  $g$  converge.

### Example 4.1

Find the Taylor series about 0 for the function  $h(z) = 2(1 - z)^{-1} - e^z$ , up to the term in  $z^3$ .

### Solution

We know that

$$(1 - z)^{-1} = 1 + z + z^2 + z^3 + \cdots, \quad \text{for } |z| < 1,$$

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots, \quad \text{for } z \in \mathbb{C}.$$

Using the Multiple Rule, we can calculate Taylor series for  $2(1 - z)^{-1}$  and  $-e^z$ , separately, and then add them using the Sum Rule to give the Taylor series for  $h(z) = 2(1 - z)^{-1} - e^z$ . However, the calculations are straightforward enough that we can do it all in one go. We obtain, by the Combination Rules,

$$\begin{aligned} h(z) &= (2 - 1) + (2 - 1)z + \left(2 - \frac{1}{2!}\right)z^2 + \left(2 - \frac{1}{3!}\right)z^3 + \cdots \\ &= 1 + z + \frac{3}{2}z^2 + \frac{11}{6}z^3 + \cdots, \quad \text{for } |z| < 1. \end{aligned}$$

The dots  $\cdots$  indicate that, even though we have been asked to find the Taylor series only up to the term in  $z^3$ , there are in fact more terms with higher powers of  $z$ .

By the uniqueness statement of Taylor's Theorem (Theorem 3.1), this power series *is* the Taylor series about 0 for  $h$ .

### Remarks

1. In examples similar to Example 4.1, we will omit the final sentence about uniqueness, and take it for granted that if a function  $f$  can be represented by a power series about  $\alpha$  that converges on an open disc centred at  $\alpha$ , then that power series *is* the Taylor series about  $\alpha$  for  $f$ .
2. For many of the Taylor series calculated in this subsection, it is difficult to find the general term, so we usually ask for Taylor series 'up to the term in  $z^3$ ' or similar, as in Example 4.1. In that particular example, instead of 'simplifying' each Taylor coefficient, we could have left the series in the form

$$h(z) = 1 + z + \left(2 - \frac{1}{2!}\right)z^2 + \left(2 - \frac{1}{3!}\right)z^3 + \cdots,$$

which has the advantage that it suggests the form of the general term of the series.

**Exercise 4.1**

Find the Taylor series about 0 for each of the following functions.

(a)  $h(z) = \text{Log}(1+z) + 3(1-z)^{-1}$  (up to the term in  $z^3$ )

(b)  $h(z) = \sin z + \cos z$  (up to the term in  $z^7$ )

Next we discuss a rule for taking the product of two power series. The procedure is similar to the procedure for multiplying two polynomials; that is, you multiply each term in the first power series by each term in the second power series, and then gather like terms. This process is encapsulated formally in the following theorem. After reading the theorem, move straight on to the example that follows it to see how the procedure works in practice – you should find it natural, and familiar.

**Theorem 4.2 Product Rule for Power Series**

Let

$$f(z) = \sum_{n=0}^{\infty} a_n(z-\alpha)^n, \quad \text{for } |z-\alpha| < R,$$

$$g(z) = \sum_{n=0}^{\infty} b_n(z-\alpha)^n, \quad \text{for } |z-\alpha| < R'.$$

If  $r = \min\{R, R'\}$ , then

$$(fg)(z) = \sum_{n=0}^{\infty} c_n(z-\alpha)^n, \quad \text{for } |z-\alpha| < r,$$

where, for each positive integer  $n$ ,

$$c_n = a_0b_n + a_1b_{n-1} + a_2b_{n-2} + \cdots + a_nb_0.$$

Thus  $c_0 = a_0b_0$ ,  $c_1 = a_0b_1 + a_1b_0$ ,  $c_2 = a_0b_2 + a_1b_1 + a_2b_0$ , and so on.

We defer a proof of the Product Rule until the end of the subsection, and instead demonstrate how it is used.

**Example 4.2**

Find the Taylor series about 0 for the function  $h(z) = (1+z)^{-2}e^z$ , up to the term in  $z^3$ .

**Solution**

Using the Taylor series for  $(1+z)^{-2}$  from Example 3.5, and the basic Taylor series for  $\exp$ , we have

$$(1+z)^{-2} = 1 - 2z + 3z^2 - 4z^3 + \cdots, \quad \text{for } |z| < 1,$$

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots, \quad \text{for } z \in \mathbb{C}.$$

Hence, for  $|z| < 1$ ,

$$h(z) = (1 - 2z + 3z^2 - 4z^3 + \cdots) \left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right).$$

To work out the Taylor series about 0 for  $h$ , we multiply each term in the left-hand bracket with each term in the right-hand bracket. For low powers of  $z$  in the Taylor series for  $h$ , we need to look at only the first few terms in each bracket. As an example, to find the  $z^2$  term in the Taylor series for  $h$ , we multiply the following terms from the left and right brackets:

$$\left(\underbrace{1}_{\text{left}} \underbrace{-2z}_{\text{left}} + \underbrace{3z^2}_{\text{left}} - 4z^3 + \cdots\right) \left(\underbrace{1}_{\text{right}} + \underbrace{z}_{\text{right}} + \underbrace{\frac{z^2}{2!}}_{\text{right}} + \frac{z^3}{3!} + \cdots\right).$$

Hence the  $z^2$  coefficient is

$$1 \times \frac{1}{2} + (-2) \times 1 + 3 \times 1 = \frac{3}{2}.$$

More generally, by the Product Rule, we have

$$\begin{aligned} h(z) &= 1 + (1 - 2)z + \frac{3}{2}z^2 + \left(\frac{1}{3!} - \frac{2}{2!} + 3 - 4\right)z^3 + \cdots \\ &= 1 - z + \frac{3}{2}z^2 - \frac{11}{6}z^3 + \cdots, \quad \text{for } |z| < 1. \end{aligned}$$

### Exercise 4.2

Find the Taylor series about 0 for each of the following functions, up to the term in  $z^3$ .

(a)  $h(z) = e^z \sin z$       (b)  $h(z) = (\cos z) \operatorname{Log}(1 + z)$

Another useful technique for manipulating Taylor series is the method of substitution, discussed briefly in Subsection 3.1. There we saw how to apply a substitution to find a Taylor series about a non-zero point by using a known Taylor series about 0. Substitutions can also be used to find the Taylor series for a function by expressing the function in a different way, again allowing us to take advantage of known Taylor series. For example, we can find the Taylor series about 0 for the function

$$f(z) = \cosh(z^2)$$

by substituting  $w = z^2$  and then using the Taylor series about 0 for the

cosh function,

$$\cosh w = 1 + \frac{w^2}{2!} + \frac{w^4}{4!} + \frac{w^6}{6!} + \cdots, \quad \text{for } w \in \mathbb{C}.$$

Substituting  $w = z^2$  into this power series gives

$$\cosh(z^2) = 1 + \frac{z^4}{2!} + \frac{z^8}{4!} + \frac{z^{12}}{6!} + \cdots, \quad \text{for } z \in \mathbb{C}.$$

We illustrate the method of substitution by three further examples.

### Example 4.3

Find the Taylor series about  $i$  for the function  $h(z) = e^z$ , up to the term in  $(z - i)^3$ .

### Solution

Our strategy is to make a substitution that allows us to find the Taylor series about  $i$  for  $h$  from a known Taylor series about 0. To do this, we first write

$$h(z) = e^z = e^{i+(z-i)} = e^i e^{z-i}.$$

Now let  $w = z - i$ , so  $h(z) = e^i e^w$ . The Taylor series about 0 for the exponential function is

$$e^w = 1 + w + \frac{w^2}{2!} + \frac{w^3}{3!} + \cdots, \quad \text{for } w \in \mathbb{C}.$$

Thus, by substituting  $w = z - i$  into this power series, and multiplying by  $e^i$ , we obtain

$$e^z = e^i e^{z-i} = e^i + e^i(z - i) + \frac{e^i}{2!}(z - i)^2 + \frac{e^i}{3!}(z - i)^3 + \cdots,$$

for  $z \in \mathbb{C}$ .

This Taylor series can also be obtained by finding a formula for the Taylor coefficients, in the manner of Subsection 3.1. After all, the exponential function is its own derivative, so

$$h^{(n)}(i) = h(i) = e^i, \quad \text{for } n = 0, 1, 2, \dots$$

It follows, then, from Taylor's Theorem (Theorem 3.1) that the general term of the Taylor series about  $i$  for  $h$  is

$$\frac{e^i}{n!}(z - i)^n,$$

which agrees with the solution to Example 4.3.

**Example 4.4**

Find the Taylor series about 0 for  $h(z) = (1 + z^2)^{-1}$ , up to the term in  $z^6$ .

**Solution**

We know the Taylor series about 0 for  $(1 - w)^{-1}$ , so we can substitute  $w = -z^2$  into this series, taking care with the radius of convergence. In more detail, recall that

$$(1 - w)^{-1} = 1 + w + w^2 + w^3 + \cdots, \quad \text{for } |w| < 1.$$

We define  $w = -z^2$ , and observe that  $|w| < 1$  if and only if  $|z| < 1$ . Thus, substituting  $w = -z^2$  into the series above, we see that

$$(1 + z^2)^{-1} = 1 - z^2 + z^4 - z^6 + \cdots, \quad \text{for } |z| < 1.$$

In accordance with Theorem 3.2(a), the Taylor series in Example 4.4 has only even powers, because  $h$  is an even function.

**Example 4.5**

Find the Taylor series about  $\pi/2$  for the function  $h(z) = \sin 2z$ , up to the term in  $(z - \pi/2)^5$ .

**Solution**

We seek to make a substitution that allows us to find the Taylor series about  $\pi/2$  for  $h$  from a known Taylor series about 0. To do this, we first write

$$h(z) = \sin 2z = \sin(2(z - \pi/2) + \pi).$$

Now let  $w = 2(z - \pi/2)$ . Then

$$\sin 2z = \sin(w + \pi) = \sin w \cos \pi + \cos w \sin \pi = -\sin w.$$

The Taylor series about 0 for  $\sin$  is

$$\sin w = w - \frac{w^3}{3!} + \frac{w^5}{5!} - \cdots, \quad \text{for } w \in \mathbb{C},$$

so

$$-\sin w = -w + \frac{w^3}{3!} - \frac{w^5}{5!} + \cdots, \quad \text{for } w \in \mathbb{C}.$$

Substituting  $w = 2(z - \pi/2)$  into this power series gives

$$\sin 2z = -2(z - \pi/2) + \frac{2^3}{3!}(z - \pi/2)^3 - \frac{2^5}{5!}(z - \pi/2)^5 + \cdots,$$

for  $z \in \mathbb{C}$ .

Again, this Taylor series can also be found by calculating the Taylor coefficients in the manner of Subsection 3.1.

The substitution methods demonstrated by these examples are summarised below.

### Substitution Rule for Power Series

The substitution

$$w = \lambda z^k, \quad \text{where } \lambda \neq 0, \quad k \in \mathbb{N},$$

changes a power series in powers of  $z$  with radius of convergence  $R$  to a power series in powers of  $w$  with radius of convergence  $\sqrt[k]{R/|\lambda|}$ .

The substitution

$$w = z + \beta - \alpha$$

changes a power series in powers of  $z - \alpha$  to a power series in powers of  $w - \beta$ , and preserves the radius of convergence.

The Taylor series for each of the functions in the next exercise could be found by determining the higher derivatives of the function at the given point and applying Taylor's Theorem (which was the procedure we adopted in Subsection 3.1). However, you should attempt the exercise by finding the required Taylor series using the Substitution Rule, following the approach of Examples 4.3–4.5.

### Exercise 4.3

For each of the following functions, find the Taylor series about the given point.

- (a)  $h(z) = (1 - z^2)^{-1/2}$  about 0 (up to the term in  $z^6$ )
- (b)  $h(z) = \cosh z$  about  $i\pi/2$  (up to the term in  $(z - i\pi/2)^5$ )
- (c)  $h(z) = z^\alpha$  about 1, where  $\alpha \in \mathbb{C}$  (up to the term in  $(z - 1)^2$ )
- (d)  $h(z) = \text{Log}(1 + z)$  about 2 (up to the term in  $(z - 2)^4$ )

The Substitution Rule is in fact just a special case of a more general rule for finding the Taylor series of a composition of two functions. This more general rule tells us that, essentially, you can compose the Taylor series of two analytic functions in the same sort of way that you compose two polynomials.

**Theorem 4.3 Composition Rule for Power Series**

Let

$$f(z) = \sum_{n=0}^{\infty} a_n(z - \alpha)^n, \quad \text{for } |z - \alpha| < R,$$

$$g(w) = \sum_{n=0}^{\infty} b_n(w - \beta)^n, \quad \text{for } |w - \beta| < R'.$$

If  $\beta = f(\alpha)$ , then, for some  $r > 0$ ,

$$g(f(z)) = \sum_{n=0}^{\infty} c_n(z - \alpha)^n, \quad \text{for } |z - \alpha| < r,$$

where, for each  $n$ , the number  $c_n$  is the coefficient of  $(z - \alpha)^n$  in

$$\sum_{k=0}^n b_k \left( \sum_{l=1}^n a_l(z - \alpha)^l \right)^k.$$

We omit the proof of the Composition Rule, which is similar in type to the proof of the Product Rule given at the end of this subsection. However, let us briefly look at where the unspecified number  $r$  comes from.

We are given power series about  $\alpha$  and  $\beta$  for  $f$  and  $g$ , respectively, so the Differentiation Rule (Theorem 2.3) tells us that  $f$  is differentiable on  $\{z : |z - \alpha| < R\}$  and  $g$  is differentiable on  $\{w : |w - \beta| < R'\}$ . Since  $f$  is continuous at  $\alpha$  (because it is differentiable at  $\alpha$ ), and  $f(\alpha) = \beta$ , we can apply the  $\varepsilon$ - $\delta$  definition of continuity (Subsection 2.2 of Unit A3) with  $\varepsilon = R'$  and  $\delta$  equal to some positive number  $r$  with  $r \leq R$  to see that

$$|z - \alpha| < r \implies |f(z) - \beta| < R'.$$

Thus we can apply the Chain Rule to see that the composite function  $z \mapsto g(f(z))$  is analytic on the disc  $\{z : |z - \alpha| < r\}$ , so it has a Taylor series on this disc.

The Composition Rule looks complicated, but in practice you should find it natural to carry out. Roughly speaking, it says that to calculate the Taylor series about  $\alpha$  for  $g(f(z))$ , you should substitute the Taylor series about  $\alpha$  for  $f$  into the Taylor series about  $\beta = f(\alpha)$  for  $g$ . And, what is more, to calculate low powers of  $(z - \alpha)$  you need to look at only the first few terms of each series. You must also pay attention to the following warning.

Make sure that you check the condition  $\beta = f(\alpha)$  when applying the Composition Rule.

The procedure is best demonstrated by an example.

**Example 4.6**

Find the Taylor series about 0 for the function  $h(z) = \text{Log}(\cos z)$ , up to the term in  $z^6$ .

**Solution**

We will apply the Composition Rule with  $f(z) = \cos z$  and  $g(w) = \text{Log } w$ . Since  $\cos 0 = 1$ , we need to find the Taylor series about 0 for  $\cos$  and the Taylor series about 1 for  $\text{Log}$ . The Taylor series about 0 for  $\cos$  is

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots, \quad \text{for } z \in \mathbb{C},$$

and the Taylor series about 1 for  $\text{Log}$  was found in Example 3.4 to be

$$\text{Log } w = (w - 1) - \frac{(w - 1)^2}{2} + \frac{(w - 1)^3}{3} - \cdots, \quad \text{for } |w - 1| < 1.$$

Next, let  $w = \cos z$ . We wish to substitute the series for  $\cos$  into that for  $\text{Log}$ , so we need to calculate  $w - 1$ , which is

$$w - 1 = \cos z - 1 = -\frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots.$$

Then, by the Composition Rule,

$$\begin{aligned} h(z) &= \text{Log}(\cos z) \\ &= (\cos z - 1) - \frac{(\cos z - 1)^2}{2} + \frac{(\cos z - 1)^3}{3} - \cdots \\ &= \left(-\frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots\right) - \frac{1}{2} \left(-\frac{z^2}{2!} + \frac{z^4}{4!} - \cdots\right)^2 \\ &\quad + \frac{1}{3} \left(-\frac{z^2}{2!} + \cdots\right)^3 + \cdots. \end{aligned}$$

We have ignored terms in each series that, when the brackets are expanded, give rise to powers higher than  $z^6$ . Let us now expand each bracket in turn to obtain

$$\begin{aligned} h(z) &= \left(-\frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots\right) - \frac{1}{2} \left(\frac{z^4}{2!2!} - \frac{2z^6}{2!4!} + \cdots\right) \\ &\quad + \frac{1}{3} \left(-\frac{z^6}{2!2!2!} + \cdots\right) + \cdots, \end{aligned}$$

again ignoring powers higher than  $z^6$ . Simplifying this expression, we obtain

$$\begin{aligned} \text{Log}(\cos z) &= -\frac{1}{2}z^2 + \left(\frac{1}{4!} - \frac{1}{8}\right)z^4 + \left(-\frac{1}{6!} + \frac{1}{2 \times 4!} - \frac{1}{24}\right)z^6 + \cdots \\ &= -\frac{1}{2}z^2 - \frac{1}{12}z^4 - \frac{1}{45}z^6 + \cdots. \end{aligned}$$

The Composition Rule tells us that this series converges for  $|z| < r$ , where  $r$  is some unspecified positive number.

**Exercise 4.4**

Find the Taylor series about 0 for  $h(z) = e^{\sin z}$ , up to the term in  $z^5$ .

**Exercise 4.5**

By applying the Composition Rule to the Taylor series about 0 for the functions

$$f(z) = 1 - \cos z \quad \text{and} \quad g(w) = \frac{1}{1-w},$$

find the Taylor series about 0 for  $h(z) = \sec z$ , up to the term in  $z^6$ .

Another way to calculate Taylor series is to use the Differentiation and Integration Rules (Theorem 2.3 and its corollary). We demonstrate how to apply the Differentiation Rule in the following example.

**Example 4.7**

Find the Taylor series about 0 for the function  $h(z) = \tan z$ , up to the term in  $z^5$ .

**Solution**

Observe that  $h(z) = \tan z$  is the derivative of the function

$$f(z) = -\operatorname{Log}(\cos z),$$

which is the negative of the function considered in Example 4.6.

Using the Multiple Rule (with multiplier  $-1$ ), we have

$$f(z) = \frac{1}{2}z^2 + \frac{1}{12}z^4 + \frac{1}{45}z^6 + \cdots, \quad \text{for } |z| < r,$$

where  $r$  is some positive number. The Differentiation Rule tells us that  $f$  is analytic on the open disc  $\{z : |z| < r\}$  and we can differentiate term by term to give

$$\tan z = f'(z) = z + \frac{1}{3}z^3 + \frac{2}{15}z^5 + \cdots, \quad \text{for } |z| < r.$$

The Taylor series for  $\tan$  found in Example 4.7 has only odd powers, in accordance with Theorem 3.2(b), because  $\tan$  is an odd function.

Next we would like to use a similar method but with the Integration Rule instead of the Differentiation Rules to determine the Taylor series about 0 for  $\tan^{-1}$ . However, so far we have not defined this inverse function, or the inverse sine function  $\sin^{-1}$ . (We have not defined  $\cos^{-1}$  either, and we will not do so, because although the procedure for defining this inverse function is similar to the others, it is complicated by the fact that unlike  $\tan$  and  $\sin$ , the function  $\cos$  does not map 0 to 0.)

To define  $\tan^{-1}$  and  $\sin^{-1}$ , it is necessary to restrict the domains of the functions  $\tan$  and  $\sin$  in such a way that the restricted functions are one-to-one, and hence have inverse functions. In Unit C2 we will show that a convenient way to do this is to restrict the domains of both  $\tan$  and  $\sin$  to the region

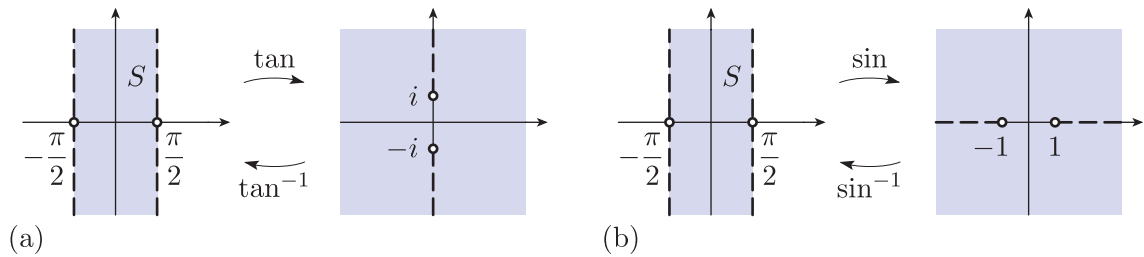
$$S = \{z : -\pi/2 < \operatorname{Re} z < \pi/2\}.$$

If we do so, then we obtain one-to-one and onto functions

$$\tan: S \longrightarrow \mathbb{C} - \{iy : y \in \mathbb{R}, |y| \geq 1\},$$

$$\sin: S \longrightarrow \mathbb{C} - \{x : x \in \mathbb{R}, |x| \geq 1\},$$

as shown in Figure 4.1. We will prove in Unit C2 that the inverse functions  $\tan^{-1}$  and  $\sin^{-1}$  are both analytic.



**Figure 4.1** The functions  $\tan$  and  $\sin$  and their inverse functions

The derivatives of  $\tan^{-1}$  and  $\sin^{-1}$  can be calculated using the Inverse Function Rule (Theorem 3.2 of Unit A4). For example, the Inverse Function Rule tells us that for  $z$  in the domain of  $\tan^{-1}$ , we have

$$(\tan^{-1})'(z) = \frac{1}{\tan' w},$$

where  $w = \tan^{-1} z$ . Now,

$$\tan' w = \sec^2 w = 1 + \tan^2 w.$$

But remember that  $\tan^2 w$  means  $(\tan w)^2$ , and since  $w = \tan^{-1} z$ , we have  $\tan w = z$ , so

$$\tan' w = 1 + (\tan w)^2 = 1 + z^2.$$

Hence

$$(\tan^{-1})'(z) = \frac{1}{1 + z^2},$$

which is the same formula as that for the derivative of the real inverse tangent function.

Similarly, it can be shown that

$$(\sin^{-1})'(z) = \frac{1}{\sqrt{1 - z^2}}.$$

The formula for  $(\tan^{-1})'(z)$  is needed in the next example.

**Example 4.8**

Find the Taylor series about 0 for the function  $h(z) = \tan^{-1} z$ , up to the term in  $z^7$ .

**Solution**

We know that

$$(\tan^{-1})'(z) = (1 + z^2)^{-1}, \quad \text{for } z \in \mathbb{C} - \{iy : y \in \mathbb{R}, |y| \geq 1\}.$$

Also, by Example 4.4,

$$(1 + z^2)^{-1} = 1 - z^2 + z^4 - z^6 + \cdots, \quad \text{for } |z| < 1.$$

The set  $\mathbb{C} - \{iy : y \in \mathbb{R}, |y| \geq 1\}$  contains the open disc  $\{z : |z| < 1\}$ , so on integrating the power series for  $(1 + z^2)^{-1}$  term by term and applying the Integration Rule, we see that

$$\tan^{-1} z = b_0 + z - \frac{1}{3}z^3 + \frac{1}{5}z^5 - \frac{1}{7}z^7 + \cdots, \quad \text{for } |z| < 1,$$

where  $b_0 \in \mathbb{C}$ . Since  $b_0 = \tan^{-1} 0 = 0$ , we obtain

$$\tan^{-1} z = z - \frac{1}{3}z^3 + \frac{1}{5}z^5 - \frac{1}{7}z^7 + \cdots, \quad \text{for } |z| < 1.$$

The function  $\tan$  is one-to-one on the set  $S = \{z : -\pi/2 < \operatorname{Re} z < \pi/2\}$ , and it is odd, so it follows from Exercise 3.4 that  $\tan^{-1}$  is also an odd function. Hence its Taylor series has only odd powers, by Theorem 3.2(b). (This gives us another way of deducing that the constant  $b_0$  is zero.)

**Exercise 4.6**

Using the formula

$$(\sin^{-1})'(z) = \frac{1}{\sqrt{1 - z^2}},$$

find the Taylor series about 0 for the function  $h(z) = \sin^{-1} z$ , up to the term in  $z^7$ .

To finish this subsection we prove the Product Rule, as promised earlier.

**Theorem 4.2 Product Rule for Power Series**

Let

$$f(z) = \sum_{n=0}^{\infty} a_n(z - \alpha)^n, \quad \text{for } |z - \alpha| < R,$$

$$g(z) = \sum_{n=0}^{\infty} b_n(z - \alpha)^n, \quad \text{for } |z - \alpha| < R'.$$

If  $r = \min\{R, R'\}$ , then

$$(fg)(z) = \sum_{n=0}^{\infty} c_n(z - \alpha)^n, \quad \text{for } |z - \alpha| < r,$$

where, for each positive integer  $n$ ,

$$c_n = a_0b_n + a_1b_{n-1} + a_2b_{n-2} + \cdots + a_nb_0.$$

**Proof** We prove the theorem with  $\alpha = 0$ . The general case then follows by substituting  $z - \alpha$  for  $z$ .

Let us calculate  $c_n$ , for some fixed positive integer  $n$ . Observe that, for  $|z| < r$ , we can write

$$f(z) = p_f(z) + z^{n+1}e_f(z) \quad \text{and} \quad g(z) = p_g(z) + z^{n+1}e_g(z),$$

where  $p_f$  and  $p_g$  are the *polynomial* functions

$$p_f(z) = \sum_{k=0}^n a_k z^k \quad \text{and} \quad p_g(z) = \sum_{k=0}^n b_k z^k,$$

and  $e_f$  and  $e_g$  are *analytic* functions on the open disc  $\{z : |z| < r\}$  given by

$$e_f(z) = \sum_{k=0}^{\infty} a_{k+n+1} z^k \quad \text{and} \quad e_g(z) = \sum_{k=0}^{\infty} b_{k+n+1} z^k.$$

These functions are analytic on the open disc  $\{z : |z| < r\}$  by the Differentiation Rule (Theorem 2.3) because they are both given by convergent power series on that disc.

(We split up  $f$  and  $g$  in this way because the terms of the Taylor series for  $f$  and  $g$  involving powers of  $z$  higher than  $n$  do not affect the value of  $c_n$ .)

For  $|z| < r$ , we have

$$(fg)(z) = p_f(z)p_g(z) + z^{n+1}e(z),$$

where

$$e(z) = p_f(z)e_g(z) + p_g(z)e_f(z) + z^{n+1}e_f(z)e_g(z).$$

The function  $e$  is analytic on  $\{z : |z| < r\}$ , since it is a combination of

functions that are analytic on that disc, so it has a Taylor series

$$e(z) = \sum_{k=0}^{\infty} d_k z^k, \quad \text{for } |z| < r.$$

Hence, for  $|z| < r$ , we have

$$(fg)(z) = p_f(z)p_g(z) + z^{n+1} \sum_{k=0}^{\infty} d_k z^k.$$

After multiplying the product  $p_f(z)p_g(z)$  of polynomials, we see that this last equation gives us a power series about 0 that represents  $fg$ . By the uniqueness of Taylor series representations, this power series must be the Taylor series about 0 for  $fg$ . The  $n$ th term  $c_n z^n$  is obtained from the product

$$p_f(z)p_g(z) = \left( \sum_{k=0}^n a_k z^k \right) \left( \sum_{k=0}^n b_k z^k \right),$$

so  $c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_n b_0$ . ■

## 4.2 The radius of convergence of a Taylor series

In the previous subsection we found the Taylor series for a number of functions. However, we did not specify the radius of convergence of these Taylor series, but simply gave a disc on which we knew that the series converged to the function concerned.

For instance, in Example 4.7 we saw that

$$\tan z = z + \frac{1}{3}z^3 + \frac{2}{15}z^5 + \cdots, \quad \text{for } |z| < r.$$

Here  $r$  is some positive number arising from the use of the Composition Rule in Example 4.6 to obtain the Taylor series about 0 for the function  $h(z) = \text{Log}(\cos z)$ .

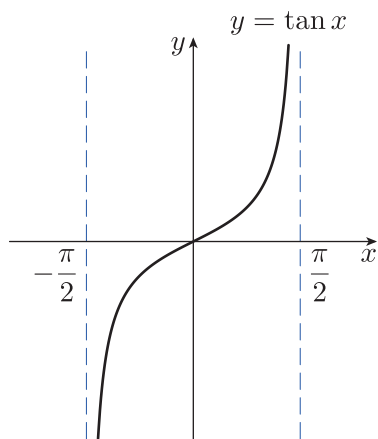
Now,  $\tan$  is analytic on its domain  $\mathbb{C} - \{(n + \frac{1}{2})\pi : n \in \mathbb{Z}\}$ , so it is analytic on the open disc  $\{z : |z| < \pi/2\}$ . Thus, by Taylor's Theorem, it can be represented on  $\{z : |z| < \pi/2\}$  by its Taylor series about 0.

Therefore we see that

$$\tan z = z + \frac{1}{3}z^3 + \frac{2}{15}z^5 + \cdots, \quad \text{for } |z| < \pi/2. \quad (4.1)$$

Note that this representation of the tangent function is *not* valid on any larger disc with centre 0, since  $\tan$  is not defined at  $\pi/2$  or  $-\pi/2$ .

Equation (4.1) shows that the radius of convergence  $R$  of the Taylor series for  $\tan$  satisfies  $R \geq \pi/2$ . It seems likely that  $R = \pi/2$ , but we cannot use the Radius of Convergence Formula (Theorem 2.2) to verify this, since we do not have a formula for the coefficients. Instead we use the following indirect argument.



**Figure 4.2** Graph of  $y = \tan x$

Suppose, in order to reach a contradiction, that  $R > \pi/2$ . Then the function

$$f(z) = z + \frac{1}{3}z^3 + \frac{2}{15}z^5 + \cdots \quad (|z| < R)$$

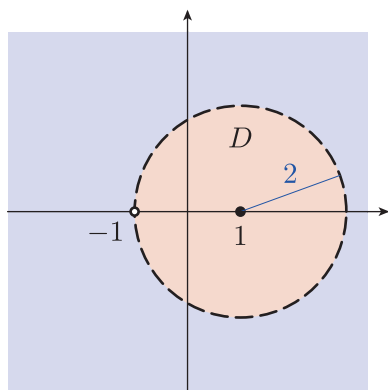
is analytic and hence continuous on  $\{z : |z| < R\}$ , so it is bounded on the compact set  $\{z : |z| \leq \pi/2\}$  by the Boundedness Theorem (Theorem 5.3 of Unit A3). This implies that  $\tan$  is bounded on  $\{z : |z| < \pi/2\}$ , which is false, as the graph of  $y = \tan x$  for  $-\pi/2 < x < \pi/2$  shown in Figure 4.2 demonstrates. Thus, contrary to our assumption,  $R$  is not greater than  $\pi/2$ , so we deduce that  $R = \pi/2$ .

The crucial fact used in this argument is that  $\tan$  is unbounded on the disc  $\{z : |z| < \pi/2\}$ , because it ‘blows up’ near the points  $\pi/2$  and  $-\pi/2$ . A similar argument can be employed to prove the following more general theorem (we omit the details).

### Theorem 4.4

Let  $f$  be a function that is analytic and unbounded on the open disc  $D = \{z : |z - \alpha| < R\}$  centred at  $\alpha$  of radius  $R$ .

Then  $D$  is the disc of convergence for the Taylor series about  $\alpha$  for  $f$ , so this Taylor series has radius of convergence  $R$ .



**Figure 4.3** The open disc  $D = \{z : |z - 1| < 2\}$  inside  $\mathbb{C} - \{-1\}$

It is important to appreciate that Theorem 4.4 can be applied only when  $f$  is *unbounded* on the open disc  $D$ .

Suppose, for example, that we wish to find the radius of convergence of the Taylor series about the point 1 for the function

$$f(z) = \frac{1}{z+1} \quad (z \in \mathbb{C} - \{-1\}).$$

Let  $D = \{z : |z - 1| < 2\}$ , as shown in Figure 4.3. This is the largest open disc centred at 1 that lies inside the domain  $\mathbb{C} - \{-1\}$ , so  $f$  is analytic on  $D$ , and  $f$  is unbounded on  $D$  because  $f(z)$  gets arbitrarily large as  $z$  approaches  $-1$  within  $D$ . By Theorem 4.4, the radius of convergence of the Taylor series about 1 for  $f$  is 2, the radius of  $D$ . We have thereby determined the radius of convergence without actually finding the Taylor series!

## Further exercises

### Exercise 4.7

Find the Taylor series about 0 for each of the following functions.

- (a)  $f(z) = (1+z)^{1/2}$  (up to the term in  $z^3$ )
- (b)  $f(z) = (1+z)^{1/2} - (1-z)^{1/2}$  (up to the term in  $z^3$ )
- (c)  $f(z) = \operatorname{Log}\left(\frac{1-z}{1+z}\right)$  (up to the term in  $z^3$ )
- (d)  $f(z) = z^3 \cos(z^2)$  (up to the term in  $z^{11}$ )
- (e)  $f(z) = (\sin z)(\cos z)$  (up to the term in  $z^5$ )
- (f)  $f(z) = \operatorname{Log}(\cosh z)$  (up to the term in  $z^6$ )
- (g)  $f(z) = \tanh z$  (up to the term in  $z^5$ )

(Hint: For part (g), use part (f).)

## 5 The Uniqueness Theorem

After working through this section, you should be able to:

- find the zeros of a function and determine their orders
- understand how to use the Uniqueness Theorem.

A significant part of this section is about analytic functions taking the same values. We say that two functions  $f$  and  $g$  **agree** on a set  $S$  if

$$f(z) = g(z), \quad \text{for all } z \in S.$$

For instance, in Unit A2 we defined the basic complex functions  $\exp$ ,  $\cos$  and  $\sin$  in such a way that they agreed with their real counterparts on the real axis. Take the complex exponential function, for example, which was defined by the formula

$$e^{x+iy} = e^x(\cos y + i \sin y).$$

This definition ensures that the complex exponential function agrees with the real exponential function on the real axis. Furthermore, when defined in this way, the complex exponential function has the useful property that it is analytic on the entire complex plane.

It is reasonable to question whether we could have defined the exponential function differently. Is there any other analytic function that agrees with the real exponential function on the real line and which could therefore have been used to define the complex exponential function?

This raises a more general question that is worthy of investigation.

If two functions  $g$  and  $h$  are analytic on a region  $\mathcal{R}$ , and  $g$  agrees with  $h$  on some subset  $S$  of  $\mathcal{R}$ , then must  $g$  and  $h$  agree on  $\mathcal{R}$ ?

Of course, the answer to this question depends on the set  $S$ . In this section we show that the answer is yes, provided that  $S$  has a limit point in  $\mathcal{R}$ . (For the definition of limit point, see Subsection 3.1 of Unit A3.)

However, the answer is, in general, no if  $S$  is a finite or infinite set without a limit point in  $\mathcal{R}$ . For example, we will see in Exercise 5.4 that the answer is no if  $\mathcal{R}$  is the entire complex plane and  $S$  is the set of positive integers.

## 5.1 Zeros of a function

We begin to answer the question posed in the box above by expressing the question in terms of the function  $f = g - h$ .

If a function  $f$  is analytic on a region  $\mathcal{R}$ , and  $f$  is zero throughout some subset  $S$  of  $\mathcal{R}$ , then must  $f$  be zero throughout  $\mathcal{R}$ ?

We use the word ‘throughout’ in this question (rather than simply ‘on’) for emphasis. A function is ‘zero throughout a set’ if it agrees with the zero function on that set. It is also said to be ‘identically zero’ on that set.

Clearly it is quite possible for  $f$  to be zero at a point  $\alpha \in \mathcal{R}$  without having to be zero throughout  $\mathcal{R}$ . So if  $S$  consists of a single point (or, more generally, a finite set of points), then the answer to the question is no. Nevertheless, it is worth pausing to consider this case a little further.

Recall that a function  $f$  has a zero at  $\alpha$  if  $f(\alpha) = 0$ . If  $f$  is analytic at  $\alpha$ , and has a zero at  $\alpha$ , then the constant term in the Taylor series about  $\alpha$  for  $f$  is zero:

$$f(z) = f^{(1)}(\alpha)(z - \alpha) + \frac{f^{(2)}(\alpha)}{2!}(z - \alpha)^2 + \frac{f^{(3)}(\alpha)}{3!}(z - \alpha)^3 + \cdots$$

When this happens, we can factor out  $(z - \alpha)$  from the series. More generally, if  $f^{(n)}(\alpha) = 0$  for all  $n$  up to but excluding  $k$ , then  $\alpha$  is a zero of  $f$  and we can now factor out  $(z - \alpha)^k$  instead of just  $(z - \alpha)$ . By analogy with polynomial factors, we then say that the zero  $\alpha$  is of *order*  $k$ .

### Definitions

Let  $f$  be a function that is analytic at  $\alpha$ . If

$$f(\alpha) = f^{(1)}(\alpha) = f^{(2)}(\alpha) = \cdots = f^{(k-1)}(\alpha) = 0, \text{ but } f^{(k)}(\alpha) \neq 0,$$

then  $f$  has a **zero of (finite) order  $k$  at  $\alpha$** .

A zero of order 1 is called a **simple zero**.

For example, the function  $\sin$  has a simple zero at  $\pi$  since

$$\sin \pi = 0 \quad \text{and} \quad \sin' \pi = \cos \pi \neq 0.$$

By contrast, the function  $f(z) = 1 + \cos z$  has a zero of order 2 at  $\pi$  since

$$f(\pi) = 0, \quad f'(\pi) = -\sin \pi = 0 \quad \text{and} \quad f''(\pi) = -\cos \pi \neq 0.$$

We usually classify the zeros of a function  $f$  without calculating its higher derivatives, by using the following theorem.

### Theorem 5.1

A function  $f$  is analytic at a point  $\alpha$ , and has a zero of order  $k$  at  $\alpha$ , if and only if, for some  $r > 0$ ,

$$f(z) = (z - \alpha)^k g(z), \quad \text{for } |z - \alpha| < r,$$

where  $g$  is a function that is analytic at  $\alpha$ , and  $g(\alpha) \neq 0$ .

Notice that if  $f(z) = (z - \alpha)^k g(z)$ , then the Taylor series about  $\alpha$  for  $f$  is equal to the Taylor series about  $\alpha$  for  $g$  multiplied by  $(z - \alpha)^k$ . Hence the Taylor series for  $f$  and  $g$  have the same radius of convergence.

**Proof** First, if  $f$  is analytic at  $\alpha$ , and has a zero of order  $k$  at  $\alpha$ , then the Taylor series about  $\alpha$  for  $f$  has the form

$$\begin{aligned} f(z) &= 0 + 0 + \cdots + 0 + \frac{f^{(k)}(\alpha)}{k!}(z - \alpha)^k + \frac{f^{(k+1)}(\alpha)}{(k+1)!}(z - \alpha)^{k+1} + \cdots \\ &= (z - \alpha)^k g(z), \end{aligned}$$

for  $|z - \alpha| < r$ , where  $r$  is some positive real number, and

$$g(z) = \sum_{n=k}^{\infty} \frac{f^{(n)}(\alpha)}{n!}(z - \alpha)^{n-k}, \quad \text{for } |z - \alpha| < r.$$

By the Differentiation Rule,  $g$  is analytic at  $\alpha$ . Furthermore,

$$g(\alpha) = f^{(k)}(\alpha)/k! \neq 0.$$

Conversely, suppose that

$$f(z) = (z - \alpha)^k g(z), \quad \text{for } |z - \alpha| < r,$$

where  $g$  is a function that is analytic at  $\alpha$ , and  $g(\alpha) \neq 0$ . Then  $g$  has a Taylor series about  $\alpha$  of the form

$$g(z) = \sum_{n=0}^{\infty} a_n(z - \alpha)^n, \quad \text{for } |z - \alpha| < r,$$

where  $a_0 \neq 0$ . But then

$$f(z) = (z - \alpha)^k g(z) = \sum_{n=0}^{\infty} a_n(z - \alpha)^{n+k}, \quad \text{for } |z - \alpha| < r.$$

Hence, by the uniqueness of Taylor series,

$$f(\alpha) = f^{(1)}(\alpha) = \cdots = f^{(k-1)}(\alpha) = 0$$

and

$$f^{(k)}(\alpha) = k! a_0 \neq 0.$$

So  $f$  has a zero of order  $k$  at  $\alpha$ . ■

This theorem tells us that we can classify the zeros of a function at  $\alpha$  by factoring out the appropriate power of  $(z - \alpha)$ , to leave an analytic function that is non-zero at  $\alpha$ , as we now illustrate.

### Example 5.1

Locate the zeros of the function

$$f(z) = (z - 2)^3(z^2 + 1)(z - i)e^z$$

and find their orders.

### Solution

The zeros of  $f$  are values of  $z$  for which  $f(z) = 0$ , so the zeros are 2,  $i$ , and  $-i$ .

To find the order of each of these zeros, we should first express the polynomial part of  $f(z)$  as a product of linear factors.

Since  $z^2 + 1 = (z - i)(z + i)$ , we obtain

$$f(z) = (z - 2)^3(z - i)^2(z + i)e^z.$$

Therefore

$$f(z) = (z - 2)^3g(z),$$

where  $g(z) = (z - i)^2(z + i)e^z$  is a function that is analytic but non-zero at 2. Hence  $f$  has a zero of order 3 at 2. Similarly,  $f$  has a zero of order 2 at  $i$ , and a simple zero at  $-i$ .

### Exercise 5.1

Locate the zeros of each of the following functions and find their orders.

- (a)  $f(z) = z^3(z - 1)^4(z + 2)$
- (b)  $f(z) = (z - 3)/(z + 2)$
- (c)  $f(z) = (z^2 + 9)^3e^{-z}/(z^2 + 4)$

If, after factoring out all the obvious  $(z - \alpha)$  terms, you are still left with a function that is zero at  $\alpha$ , then you can deal with the remaining function by finding its Taylor series about  $\alpha$ .

**Example 5.2**

Find the order of the zero of the function  $f(z) = (z - 1) \operatorname{Log} z$  at 1.

**Solution**

Although we can factor out  $(z - 1)$ , that still leaves the function  $\operatorname{Log} z$ , which is zero at 1. So, using the Taylor series about 1 for  $\operatorname{Log} z$  obtained in Example 3.4, we have

$$\begin{aligned} f(z) &= (z - 1) \operatorname{Log} z \\ &= (z - 1) \left( (z - 1) - \frac{(z - 1)^2}{2} + \frac{(z - 1)^3}{3} - \cdots \right) \\ &= (z - 1)^2 \left( 1 - \frac{z - 1}{2} + \frac{(z - 1)^2}{3} - \cdots \right) \\ &= (z - 1)^2 g(z), \end{aligned}$$

for  $|z - 1| < 1$ , where

$$g(z) = 1 - \frac{z - 1}{2} + \frac{(z - 1)^2}{3} - \cdots,$$

which is analytic and non-zero at 1. Thus  $f$  has a zero of order 2 at 1.

**Exercise 5.2**

Find the order of the zero at 0 of each of the following functions.

- (a)  $f(z) = z^4 \sin 2z$
- (b)  $f(z) = z^2(\cos z - 1)$
- (c)  $f(z) = 6 \sin(z^2) + z^2(z^4 - 6)$

Earlier we pointed out that a function that is analytic on a region  $\mathcal{R}$  can have a zero in  $\mathcal{R}$  without having to be zero throughout  $\mathcal{R}$ . The following theorem shows that this is the case only if the zero is of finite order.

**Theorem 5.2**

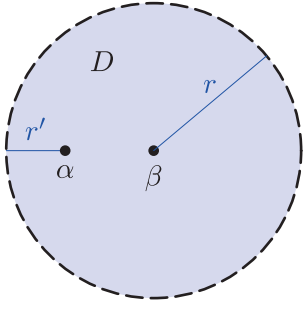
Let  $f$  be a function that is analytic on a region  $\mathcal{R}$  and not identically zero on  $\mathcal{R}$ . Then any zero of  $f$  is of finite order.

You may wish to omit the following proof on a first reading.

**Proof** Suppose that  $f$  is an analytic function on the region  $\mathcal{R}$ , and it has a zero  $\alpha$  that is not of finite order. Then  $f^{(n)}(\alpha) = 0$ , for  $n = 0, 1, 2, \dots$ . We will prove that  $f(z) = 0$  for all  $z \in \mathcal{R}$ , so  $f$  is identically zero on  $\mathcal{R}$ .

First consider the special case when  $\mathcal{R}$  is an open disc, say

$$D = \{z : |z - \beta| < r\}.$$



**Figure 5.1** Distances from points in  $D$  to the boundary of  $D$

If  $\alpha \in D$ , then the distance  $r' > 0$  from  $\alpha$  to the boundary of  $D$  is  $r' = r - |\beta - \alpha|$  (see Figure 5.1). We can construct a finite sequence of open discs

$$D_k = \{z : |z - \alpha_k| < r'\}, \quad k = 0, 1, 2, \dots, m,$$

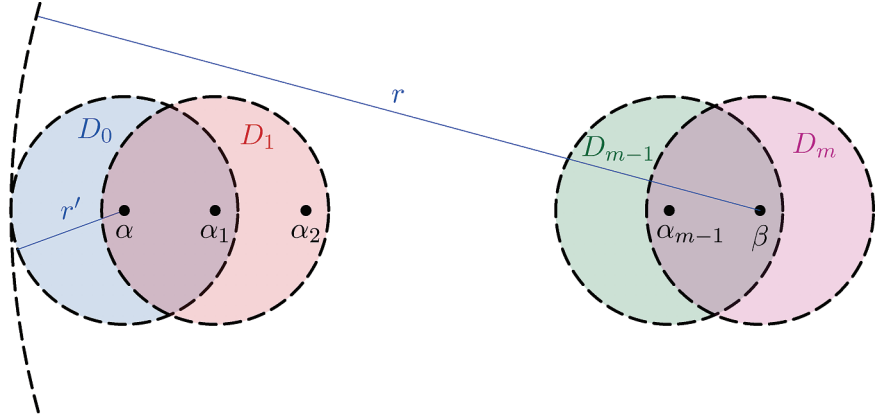
where

$$\alpha_k = \alpha + \frac{k}{m}(\beta - \alpha), \quad k = 0, 1, 2, \dots, m,$$

and  $m$  is so large that

$$\frac{|\beta - \alpha|}{m} < r'.$$

Then  $\alpha_0 = \alpha$ ,  $\alpha_m = \beta$  and, for  $k = 1, 2, \dots, m$ , the centre of the disc  $D_k$  lies in  $D_{k-1}$  (see Figure 5.2).



**Figure 5.2** Open discs  $D_0, D_1, \dots, D_{m-1}, D_m$  covering the points  $\alpha_1, \alpha_2, \dots, \alpha_{m-1}$  and  $\alpha$  and  $\beta$

All the coefficients of the Taylor series about  $\alpha = \alpha_0$  for  $f$  are zero, so

$$f(z) = 0, \quad \text{for } z \in D_0.$$

Thus  $f$  coincides with the constant function with value 0 throughout  $D_0$ , so  $f^{(n)}(z) = 0$  for  $z \in D_0$  and  $n = 0, 1, 2, \dots$ . Since  $\alpha_1 \in D_0$ , we see that

$$f^{(n)}(\alpha_1) = 0, \quad \text{for } n = 0, 1, 2, \dots$$

Repeating this argument in  $D_1, D_2, \dots, D_{m-1}$ , in turn, we deduce that

$$f^{(n)}(\beta) = 0, \quad \text{for } n = 0, 1, 2, \dots$$

Thus, using the Taylor series about  $\beta$  for  $f$ , we find that

$$f(z) = 0, \quad \text{for } z \in D,$$

as required.

Now let  $\mathcal{R}$  be a general region and suppose that  $\beta \in \mathcal{R}$  with  $\beta \neq \alpha$ . Then we can join  $\alpha$  to  $\beta$  by a path  $\Gamma : \gamma(t)$  ( $t \in [a, b]$ ) lying in  $\mathcal{R}$  (so  $\gamma(a) = \alpha$ ,  $\gamma(b) = \beta$ ) and then apply the Paving Theorem (Theorem 3.7 of Unit B1).

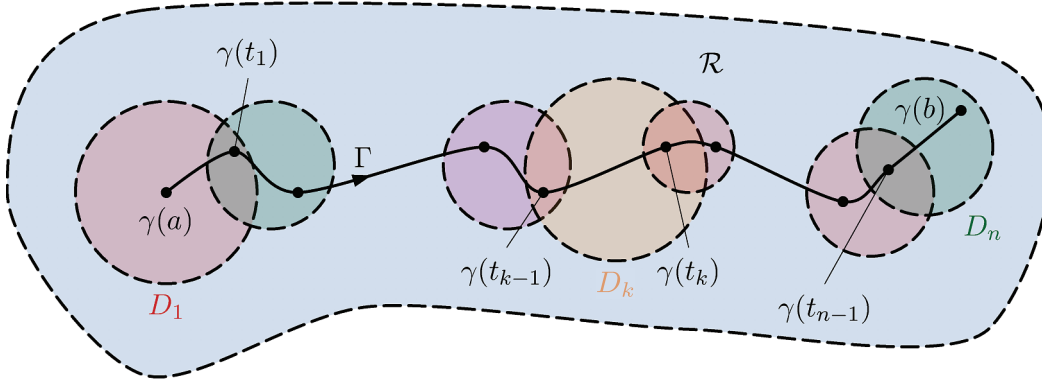
This provides a finite sequence of discs  $D_1, D_2, \dots, D_n$  in  $\mathcal{R}$  and a finite sequence of points  $t_0, t_1, \dots, t_n$ , with

$$a = t_0 < t_1 < \dots < t_n = b,$$

such that

$$\gamma([t_{k-1}, t_k]) \subseteq D_k, \quad \text{for } k = 1, 2, \dots, n,$$

as illustrated in Figure 5.3.



**Figure 5.3** Open discs  $D_1, D_2, \dots, D_n$  paving a path  $\Gamma$

Since  $f^{(n)}(\alpha) = 0$ , for  $n = 0, 1, 2, \dots$ , and  $\alpha = \gamma(t_0) \in D_1$ , it follows, by the special case above, that  $f(z) = 0$ , for  $z \in D_1$ , and hence that

$$f^{(n)}(\gamma(t_1)) = 0, \quad \text{for } n = 0, 1, 2, \dots$$

Repeating this argument with  $D_2, D_3, \dots, D_n$ , in turn, we deduce that  $f(\gamma(t_n)) = f(\beta) = 0$ . Since  $\beta \in \mathcal{R}$  was arbitrary, the proof is complete. ■

We are now well on the way to answering the question posed at the beginning of this subsection. For if  $f$  is analytic on a region  $\mathcal{R}$  and has a zero in  $\mathcal{R}$  that is not of finite order, then  $f$  must be identically zero on  $\mathcal{R}$ .

But how do we recognise a zero that is not of finite order? To help us do this, we make the following definition.

### Definition

A zero  $\alpha$  of a function  $f$  is said to be **isolated** if there is a disc centred at  $\alpha$  that contains no other zeros of  $f$ .

We then make the following observation.

### Theorem 5.3 Isolated Zeros

A zero of finite order is isolated.

**Proof** Suppose that  $\alpha$  is a zero of  $f$  of order  $k$ . Then, by Theorem 5.1,

$$f(z) = (z - \alpha)^k g(z),$$

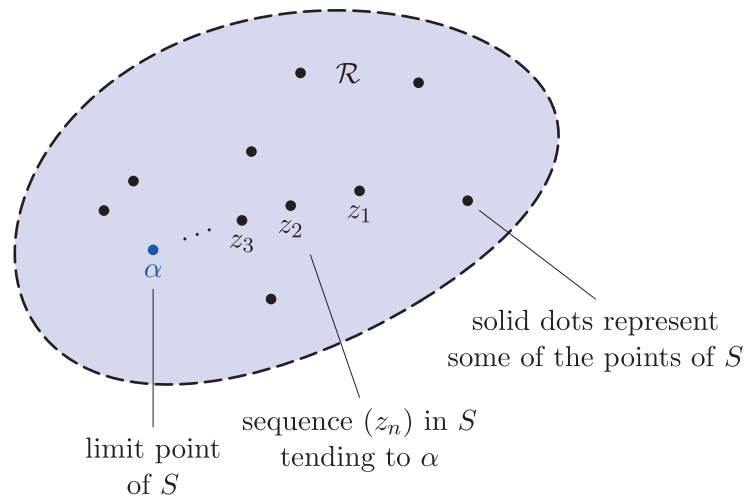
where  $g$  is analytic at  $\alpha$ , and  $g(\alpha) \neq 0$ . Since  $g$  is analytic at  $\alpha$ , it must be continuous at  $\alpha$ , so  $g$  is non-zero on some open disc with centre  $\alpha$ . Thus  $f$  is non-zero on the same open disc with centre  $\alpha$ , except at  $\alpha$  itself. Hence  $\alpha$  is an isolated zero. ■

Now, a zero cannot be isolated if it is the limit point of a set of zeros. We can therefore answer the question posed at the beginning of this subsection, as follows.

### Theorem 5.4

Let  $f$  be a function that is analytic on a region  $\mathcal{R}$ , and let  $S$  be a set of zeros of  $f$  in  $\mathcal{R}$  that has a limit point in  $\mathcal{R}$ . Then  $f$  is identically zero on  $\mathcal{R}$ .

**Proof** Let  $(z_n)$  be a sequence of zeros in  $S$  (see Figure 5.4) that converges to a limit point  $\alpha$  of  $S$  in  $\mathcal{R}$ . Since  $f$  is continuous at  $\alpha$ , it follows that  $f(\alpha) = \lim_{n \rightarrow \infty} f(z_n) = 0$ , so  $\alpha$  is a zero of  $f$  in  $\mathcal{R}$ . Furthermore, the zero  $\alpha$  is not isolated because in *any* disc centred at  $\alpha$  there is some zero  $z_n$ . By Theorem 5.3,  $\alpha$  is not of finite order. But Theorem 5.2 tells us that if  $f$  is not identically zero on  $\mathcal{R}$ , then a zero of  $f$  such as  $\alpha$  must be of finite order. We deduce that  $f$  is identically zero on  $\mathcal{R}$ . ■



**Figure 5.4** A sequence  $(z_n)$  in  $S$  converging to a limit point  $\alpha$  in  $\mathcal{R}$

An immediate consequence of this theorem is the Uniqueness Theorem.

### Theorem 5.5 Uniqueness Theorem

Let  $f$  and  $g$  be functions that are analytic on a region  $\mathcal{R}$ , and suppose that  $f$  agrees with  $g$  throughout a subset  $S$  of  $\mathcal{R}$ , where  $S$  has a limit point in  $\mathcal{R}$ . Then  $f$  agrees with  $g$  throughout  $\mathcal{R}$ .

We can now address a question from the start of this section – whether there is an analytic function other than the complex exponential function that agrees with the real exponential function on the real line.

### Example 5.3

Prove that the function

$$f(x + iy) = e^x(\cos y + i \sin y),$$

is the unique entire function that agrees with the real exponential function on  $\mathbb{R}$ .

### Solution

Suppose that  $g$  is an entire function that agrees with the real exponential function on  $\mathbb{R}$ . We apply the Uniqueness Theorem with  $S = \mathbb{R}$  and  $\mathcal{R} = \mathbb{C}$ . Since  $\mathbb{R}$  has a limit point in  $\mathbb{C}$  (for example, 0) and

$$g(x) = e^x = f(x), \quad \text{for } x \in \mathbb{R},$$

it follows from the Uniqueness Theorem that

$$g(z) = e^z = f(z), \quad \text{for } z \in \mathbb{C}.$$

### Exercise 5.3

Show that if the functions  $f$  and  $g$  are analytic on a region  $\mathcal{R}$  and are represented by the same Taylor series on an open disc  $D \subseteq \mathcal{R}$ , then  $f$  agrees with  $g$  on  $\mathcal{R}$ .

To apply Theorem 5.5,  $S$  does not have to be the real line or an open disc: any set with a limit point in  $\mathcal{R}$  will do.

### Exercise 5.4

Let  $f$  and  $g$  be entire functions. Determine which of the following conditions are sufficient to ensure that  $f$  and  $g$  are equal.

- (a)  $f$  and  $g$  agree on the set  $S = \{z : |z| = 2\}$ .
- (b)  $f$  and  $g$  agree on the set of positive integers  $S = \mathbb{N}$ .
- (c)  $f$  and  $g$  agree on the set  $S = \{1/n : n \in \mathbb{N}\}$ .

**Exercise 5.5**

Prove that if  $f$  is an entire function and

$$f(i/n) = -1/n^2, \quad \text{for } n = 1, 2, \dots,$$

then

$$f(z) = z^2, \quad \text{for } z \in \mathbb{C}.$$

**5.2 Using power series to define functions**

You may wish to omit this subsection on a first reading.

We began this section with a reminder about the way we used real functions to define the basic complex functions  $\exp$ ,  $\sin$  and  $\cos$ . However, there is an alternative way to define these basic functions that avoids the need to use real functions at all, namely to give a definition in terms of power series. We devote this final subsection to a brief summary of this alternative approach, which is adopted in many complex analysis texts.

Starting with the exponential function, we *define*

$$\exp z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \dots \quad (z \in \mathbb{C}).$$

Since this power series has an infinite radius of convergence, we know that this function is entire.

Having defined the exponential function in this way, we can check that it has all the properties we wish it to have. For example, using the Differentiation Rule, we can differentiate the power series term by term and check that the exponential function is its own derivative:

$$\exp' = \exp.$$

Next we can find the Taylor series for  $\exp$  about any point  $\alpha$ . Since

$$\exp \alpha = \exp' \alpha = \exp'' \alpha = \dots,$$

we see from Taylor's Theorem that

$$\begin{aligned} \exp z &= \exp \alpha \left( 1 + (z - \alpha) + \frac{(z - \alpha)^2}{2!} + \frac{(z - \alpha)^3}{3!} + \dots \right) \\ &= \exp \alpha \exp(z - \alpha). \end{aligned}$$

On substituting  $z = \alpha + \beta$ , we obtain the exponential addition rule:

$$\exp(\alpha + \beta) = \exp \alpha \exp \beta, \quad \text{for } \alpha, \beta \in \mathbb{C}.$$

The trigonometric and hyperbolic functions can then be defined in terms of the exponential function as they were earlier, in Unit A2:

$$\sin z = \frac{\exp(iz) - \exp(-iz)}{2i}, \quad \cos z = \frac{\exp(iz) + \exp(-iz)}{2},$$

$$\sinh z = \frac{\exp z - \exp(-z)}{2}, \quad \cosh z = \frac{\exp z + \exp(-z)}{2},$$

and

$$\tan z = \frac{\sin z}{\cos z}, \quad \tanh z = \frac{\sinh z}{\cosh z}.$$

The trigonometric and hyperbolic identities then follow as in Section 4 of Unit A2. Also, the derivatives of the trigonometric and hyperbolic functions are found as in Subsection 3.1 of Unit A4. Since the same derivatives are obtained as in our original treatment, the Taylor series must be the same as well.

With more work we can define the function  $\text{Log}$  to be the inverse of the exponential function restricted to the strip  $\{z : -\pi < \text{Im } z \leq \pi\}$ , and we can use  $\text{Log}$  to define the principal power functions.

In summary, we have indicated how the basic functions of complex analysis can be defined starting with the power series definition of the exponential function.

Of course, we need not restrict ourselves to the exponential function: Taylor's Theorem tells us that *any* analytic function can be represented by a power series (on some disc).

Many analytic functions are not readily expressible in terms of the basic complex functions, and it is often easier to specify such functions in terms of power series. For example, the analytic solutions of some differential equations are best found by substituting into the equation an arbitrary power series (using the Differentiation Rule for the derivatives). We can then equate the coefficients of corresponding powers of  $z$  and hence obtain a recurrence relation for the coefficients of the power series. This series gives a solution of the equation that is valid on the disc of convergence of the series.

Although a power series determines a function only on the disc of convergence of the series, the Uniqueness Theorem ensures that this is sufficient to determine an analytic function throughout its domain, providing its domain is a region. Later in the module we will show how we can extend the specification of a function on a disc to the whole of its domain by using a technique known as *analytic continuation*.

## Further exercises

### Exercise 5.6

Locate the zeros of each of the following functions and find their orders.

(a)  $f(z) = z^2(z^2 + 4)^3$       (b)  $f(z) = z \sin z$

(Hint: For part (b), use the formula

$$\sin(z - k\pi) = (-1)^k \sin z, \quad \text{for } k \in \mathbb{Z},$$

which can be proved using the addition formula for  $\sin$ .)



Brook Taylor

### History of Taylor series

The first published accounts of Taylor series for real functions appeared in the work of the Kerala School of Astronomy and Mathematics, based in Kerala, Southern India, between the fourteenth and sixteenth centuries. The School discovered Taylor series for trigonometric functions, with proofs, which they published in Sanskrit verse. The following passage is translated from the commentary *Yuktibhāṣā* (c.1530), written by the Indian astronomer and mathematician Jyeṣṭhadeva (c.1500–c.1575). Remarkably, it describes in words the Taylor series for the inverse tangent function obtained in Example 4.8, even including conditions for convergence. Jyeṣṭhadeva credits the discovery to the Indian astronomer and mathematician Madhava (c.1340–c.1425), who is considered to be the founder of the Kerala School.

The product of the given sine and the radius, divided by the cosine, is the first result. From the first [and the second, third, ...] results, obtain [successively] a sequence of results by taking repeatedly the square of the sine as the multiplier and the square of the cosine as the divisor. Divide the above results in order by the odd numbers one, three, ... [to get the full sequence of terms]. From the sum of the odd terms subtract the sum of the even terms. The result becomes the arc. In this connection ... the sine of the arc or that of its complement, whichever is smaller, should be taken here [as the given sine]; otherwise the terms obtained by the repeated process will not tend to the vanishing magnitude.

(Gupta, 1973, p. 67–70, cited in Katz, 1993, p. 451)

Much later, mathematicians in Europe such as Isaac Newton and the Scottish mathematician James Gregory (1638–1675) used Taylor series for a variety of functions, including logarithms. The most significant advance came in 1715, when the English mathematician Brook Taylor (1685–1731) published the result that we now call Taylor's Theorem, for real functions, although without any discussion of convergence. Taylor series were introduced to complex analysis by Cauchy, and he even used them to obtain a version of the Uniqueness Theorem, as we have done. In 1845, he wrote:

Suppose that two (continuous) functions of  $x$  are always equal to each other for values of  $x$  close to a given value. Then on varying  $x$  by 'insensible' degrees the functions remain equal to each other for as long as they remain continuous.

(Bottazzini and Gray, 2013, p. 182)

Later, in 1851, Cauchy realised that it was not continuity but analyticity that was needed to make a theorem such as the Uniqueness Theorem work.

# Solutions to exercises

## Solution to Exercise 1.1

We have

$$\begin{aligned}s_0 &= 1, \\s_1 &= 1 + \frac{i}{2}, \\s_2 &= 1 + \frac{i}{2} + \left(\frac{i}{2}\right)^2 = \frac{3}{4} + \frac{i}{2}, \\s_3 &= 1 + \frac{i}{2} + \left(\frac{i}{2}\right)^2 + \left(\frac{i}{2}\right)^3 = \frac{3}{4} + \frac{3i}{8}.\end{aligned}$$

## Solution to Exercise 1.2

(a) Here

$$\begin{aligned}s_n &= 7((-i) + (-i)^2 + \cdots + (-i)^n) \\&= -7i(1 + (-i) + (-i)^2 + \cdots + (-i)^{n-1}) \\&= -7i\left(\frac{1 - (-i)^n}{1 - (-i)}\right).\end{aligned}$$

Since  $(-i)^{4n} = 1$  and  $(-i)^{4n+1} = -i$ , for  $n = 1, 2, \dots$ , it follows that

$$s_{4n} = 0 \quad \text{and} \quad s_{4n+1} = -7i, \quad \text{for } n = 1, 2, \dots$$

Hence  $(s_n)$  diverges, by the First Subsequence Rule (Theorem 1.6(a) of Unit A3), so the series diverges.

(b) To simplify the algebra, let us define

$\alpha = (1 - i)/2$ . Then

$$\begin{aligned}s_n &= \alpha + \alpha^2 + \cdots + \alpha^n \\&= \alpha(1 + \alpha + \cdots + \alpha^{n-1}) \\&= \alpha\left(\frac{1 - \alpha^n}{1 - \alpha}\right) \\&= \frac{(1 - i)/2}{1 - (1 - i)/2} \left(1 - \left(\frac{1 - i}{2}\right)^n\right) \\&= \left(\frac{1 - i}{1 + i}\right) \left(1 - \left(\frac{1 - i}{2}\right)^n\right).\end{aligned}$$

Since  $|\alpha| < 1$ , the sequence  $(\alpha^n)$  converges to 0. Hence

$$\lim_{n \rightarrow \infty} s_n = \frac{1 - i}{1 + i} = \frac{(1 - i)(1 - i)}{2} = -i,$$

so the series converges to  $-i$ .

## Solution to Exercise 1.3

(a) The sequence  $((1 + i)^n)$  is not null because each term has modulus greater than 1. Hence

$$\sum_{n=1}^{\infty} (1 + i)^n$$

diverges, by the Non-null Test.

(b) The sequence  $(i(-1)^n)$  is not null because each term has modulus 1. Hence

$$\sum_{n=1}^{\infty} i(-1)^n$$

diverges, by the Non-null Test.

(c) By the Combination Rules for sequences,

$$\frac{n^2 + i}{2n^2 + n + 3} = \frac{1 + i/n^2}{2 + 1/n + 3/n^2} \rightarrow \frac{1}{2} \text{ as } n \rightarrow \infty.$$

Hence

$$\sum_{n=1}^{\infty} \frac{n^2 + i}{2n^2 + n + 3}$$

diverges, by the Non-null Test.

## Solution to Exercise 1.4

(a) The series diverges, since  $|-i| = 1$ .

(b) The series converges, since  $|(1 - i)/2| < 1$ . The sum is

$$\frac{(1 - i)/2}{1 - (1 - i)/2} = \frac{1 - i}{1 + i} = \frac{(1 - i)(1 - i)}{2} = -i.$$

## Solution to Exercise 1.5

Let  $(s_n)$  be the sequence of partial sums given by

$$s_n = \sum_{k=1}^n z_k,$$

and let

$$s = \sum_{k=1}^{\infty} z_k.$$

Then  $s_n \rightarrow s$ , so, by the Multiple Rule for sequences,  $\lambda s_n \rightarrow \lambda s$ . But

$$\lambda s_n = \sum_{k=1}^n \lambda z_k,$$

so we see that  $\sum_{k=1}^{\infty} \lambda z_k$  converges to  $\lambda s$ . So

$$\sum_{k=1}^{\infty} \lambda z_k = \lambda s = \lambda \sum_{k=1}^{\infty} z_k.$$

## Solution to Exercise 1.6

Here

$$\cos nx = \operatorname{Re}(e^{inx}), \quad \text{for } n = 0, 1, 2, \dots$$

So, by Theorems 1.2 and 1.5, and the calculation in Example 1.2,

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{1}{2^n} \cos nx &= \operatorname{Re} \left( \sum_{n=0}^{\infty} \frac{1}{2^n} e^{inx} \right) \\ &= \operatorname{Re} \left( \frac{(1 - \frac{1}{2} \cos x) + \frac{1}{2} i \sin x}{\frac{5}{4} - \cos x} \right) \\ &= \frac{4 - 2 \cos x}{5 - 4 \cos x}. \end{aligned}$$

## Solution to Exercise 1.7

Here

$$\left| \frac{\cos n}{n\sqrt{n}} \right| \leq \frac{1}{n^{3/2}}, \quad \text{for } n = 1, 2, \dots$$

Furthermore,  $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$  converges, by Theorem 1.3, so

$$\sum_{n=1}^{\infty} \frac{\cos n}{n\sqrt{n}}$$

converges, by the Comparison Test.

## Solution to Exercise 1.8

Let  $a_n = |z_n|$ . Then  $\sum_{n=1}^{\infty} a_n$  converges.

Furthermore,  $|z_n| \leq a_n$ , for  $n = 1, 2, \dots$

So, by the Comparison Test,  $\sum_{n=1}^{\infty} z_n$  converges.

## Solution to Exercise 1.9

(a) In this case

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n},$$

which diverges, by Theorem 1.3. It follows that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

is *not* absolutely convergent.

(b) Here

$$\sum_{n=0}^{\infty} \left| \frac{(-1)^n (1+i)^n}{2^n} \right| = \sum_{n=0}^{\infty} \left( \frac{1}{\sqrt{2}} \right)^n$$

is a convergent geometric series. Thus

$$\sum_{n=0}^{\infty} \frac{(-1)^n (1+i)^n}{2^n}$$

is absolutely convergent.

## Solution to Exercise 1.10

Let  $(s_n)$  be the sequence of partial sums for the series. Then  $s_{2n}$  is equal to

$$\left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \dots + \left(\frac{1}{2n-1} - \frac{1}{2n}\right).$$

Observe that, for  $k = 1, 2, \dots$ ,

$$0 \leq \left(\frac{1}{k} - \frac{1}{(k+1)}\right) = \frac{1}{k(k+1)} \leq \frac{1}{k^2}.$$

Therefore

$$1 - \frac{1}{2} \leq 1,$$

$$\frac{1}{3} - \frac{1}{4} \leq \frac{1}{3^2},$$

$\vdots$

$$\frac{1}{2n-1} - \frac{1}{2n} \leq \frac{1}{(2n-1)^2},$$

so

$$s_{2n} \leq 1 + \frac{1}{3^2} + \dots + \frac{1}{(2n-1)^2}.$$

Since  $\sum_{k=1}^{\infty} \frac{1}{k^2}$  converges, we deduce that the

sequence  $(s_{2n})$  is bounded above. It is an increasing sequence, so it must converge to a limit  $\alpha$ , by the Monotone Convergence Theorem.

Furthermore,

$$s_{2n+1} = s_{2n} + \frac{1}{2n+1},$$

and  $(1/(2n+1))$  is a null sequence, so  $(s_{2n+1})$  converges to  $\alpha$  also.

Therefore  $(s_n)$  converges to  $\alpha$  as well (by Exercise 1.14 of Unit A3).

## Solution to Exercise 1.11

(a) Let  $z_n = \frac{n^2}{3^n + i}$ . Then

$$\begin{aligned} \left| \frac{z_{n+1}}{z_n} \right| &= \left| \frac{(n+1)^2}{3^{n+1} + i} \right| \bigg/ \left| \frac{n^2}{3^n + i} \right| \\ &= \left| \frac{3^n + i}{3^{n+1} + i} \right| \left( \frac{n+1}{n} \right)^2 \\ &= \left| \frac{1 + i/3^n}{3 + i/3^n} \right| \left( 1 + \frac{1}{n} \right)^2. \end{aligned}$$

Using the Combination Rules for sequences, and the fact that  $z \mapsto |z|$  is a continuous function, we see that  $|z_{n+1}/z_n| \rightarrow 1/3$  as  $n \rightarrow \infty$ . It follows

from the Ratio Test that  $\sum_{n=1}^{\infty} \frac{n^2}{3^n + i}$  is absolutely convergent.

(b) The series converges absolutely if  $z = 0$ , so let us suppose that  $z \neq 0$ . Let  $z_n = \frac{z^n}{n!}$ . Then

$$\left| \frac{z_{n+1}}{z_n} \right| = \left| \frac{z^{n+1}}{(n+1)!} \right| \bigg/ \left| \frac{z^n}{n!} \right| = \frac{|z|}{n+1},$$

which tends to 0 as  $n \rightarrow \infty$ . It follows from the

Ratio Test that  $\sum_{n=0}^{\infty} \frac{z^n}{n!}$  is absolutely convergent (and hence convergent) for all  $z \in \mathbb{C}$ .

## Solution to Exercise 1.12

(a) We have  $s_0 = i$ ,  $s_1 = i + i = 2i$ ,  $s_2 = 3i$  and  $s_3 = 4i$ . In general,  $s_n = (n+1)i$ ,  $n = 0, 1, 2, \dots$

(b) We have  $s_0 = i$ ,

$$s_1 = i + \frac{i}{10} = 1.1i,$$

$$s_2 = 1.1i + \frac{i}{100} = 1.11i,$$

$$s_3 = 1.11i + \frac{i}{1000} = 1.111i.$$

In general,

$$\begin{aligned} s_n &= i \left( \frac{1 - (1/10)^{n+1}}{1 - (1/10)} \right) \\ &= \frac{i}{9} \left( \frac{10^{n+1} - 1}{10^n} \right), \quad \text{for } n = 0, 1, 2, \dots \end{aligned}$$

In decimal form,  $s_n$  can be written as  $1.11\dots 1i$ , with  $n$  1s after the decimal point.

(c) We have  $s_0 = 1$ ,  $s_1 = 1 + i$ ,  $s_2 = 1 + i - 1 = i$  and  $s_3 = 1 + i - 1 - i = 0$ .

In general,

$$s_n = \begin{cases} 1, & n = 0, 4, 8, \dots, \\ 1 + i, & n = 1, 5, 9, \dots, \\ i, & n = 2, 6, 10, \dots, \\ 0, & n = 3, 7, 11, \dots \end{cases}$$

## Solution to Exercise 1.13

Recall from Theorem 1.2 that, for  $a \neq 0$ , the geometric series

$$\sum_{n=0}^{\infty} az^n$$

converges to  $a/(1-z)$  if  $|z| < 1$ , and it diverges if  $|z| \geq 1$ .

(a) The series  $\sum_{n=0}^{\infty} \frac{1}{\sqrt{2}}(1-i)^n$  diverges because it is a geometric series with  $a = 1/\sqrt{2}$  and  $z = 1-i$ , where  $|z| = |1-i| = \sqrt{2} > 1$ .

(b) The series  $\sum_{n=0}^{\infty} \left( \frac{1+i}{2} \right)^n$  converges because it is a geometric series with  $a = 1$  and  $z = (1+i)/2$ , where  $|z| = |(1+i)/2| = \sqrt{2}/2 < 1$ .

Also,

$$\begin{aligned} \sum_{n=0}^{\infty} \left( \frac{1+i}{2} \right)^n &= \frac{1}{1 - (1+i)/2} \\ &= \frac{2}{1-i} = 1+i. \end{aligned}$$

(c) The series  $\sum_{n=0}^{\infty} \left( \frac{1-i}{\sqrt{2}} \right)^n$  diverges because it is a geometric series with  $a = 1$  and  $z = (1-i)/\sqrt{2}$ , where  $|z| = |(1-i)/\sqrt{2}| = 1$ .

(d) The series  $\sum_{n=2}^{\infty} \binom{n}{2} i^n$  diverges by the Non-null Test, because the terms

$$\binom{n}{2} i^n = \frac{n(n-1)}{2} i^n$$

do not form a null sequence.

### Solution to Exercise 1.14

First note that

$$\begin{aligned} 2^{-n/2} \cos \frac{n\pi}{4} &= \operatorname{Re} \left( 2^{-n/2} \exp \left( \frac{n\pi}{4} i \right) \right) \\ &= \operatorname{Re} \left( \frac{1}{\sqrt{2}} e^{\pi i/4} \right)^n \\ &= \operatorname{Re} \left( \frac{1+i}{2} \right)^n. \end{aligned}$$

But, by Exercise 1.13(b),

$$\sum_{n=0}^{\infty} \left( \frac{1+i}{2} \right)^n = 1+i,$$

so

$$\sum_{n=0}^{\infty} 2^{-n/2} \cos \frac{n\pi}{4} = \operatorname{Re}(1+i) = 1,$$

by Theorem 1.5.

### Solution to Exercise 1.15

(a) The series  $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$  is absolutely convergent, by the Comparison Test, as  $|1/(n^2+1)| \leq 1/n^2$ , for  $n = 1, 2, \dots$ , and  $\sum_{n=1}^{\infty} 1/n^2$  converges by Theorem 1.3.

(b) The series  $\sum_{n=1}^{\infty} \frac{1}{n^2+i}$  is absolutely convergent, by the Comparison Test. Indeed,

$$|1/(n^2+i)| = 1/\sqrt{n^4+1} \leq 1/n^2,$$

for  $n = 1, 2, \dots$ , and  $\sum_{n=1}^{\infty} 1/n^2$  converges by Theorem 1.3.

(c) The series  $\sum_{n=1}^{\infty} \frac{2^n - i}{n^2}$  diverges. Indeed,

with  $z_n = (2^n - i)/n^2$ , we have

$$\begin{aligned} \left| \frac{z_{n+1}}{z_n} \right| &= \left| \frac{2^{n+1} - i}{2^n - i} \right| \times \left| \frac{n^2}{(n+1)^2} \right| \\ &= \left| \frac{2 - i/2^n}{1 - i/2^n} \right| \times \left| \frac{1}{(1 + 1/n)^2} \right| \rightarrow 2 \end{aligned}$$

as  $n \rightarrow \infty$ , so the series diverges by the Ratio Test.

(d) The series  $\sum_{n=1}^{\infty} \frac{i^n}{n\sqrt{n}}$  is absolutely convergent,

by the Comparison Test, as  $\left| \frac{i^n}{n\sqrt{n}} \right| = \frac{1}{n^{3/2}}$ , for

$n = 1, 2, \dots$ , and  $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$  converges by

Theorem 1.3.

(e) The series  $\sum_{n=0}^{\infty} e^{n(i-1)}$  is absolutely convergent,

by the Ratio Test. Indeed, with  $z_n = e^{n(i-1)}$ , we have

$$\begin{aligned} \left| \frac{z_{n+1}}{z_n} \right| &= \left| e^{(n+1)(i-1)-n(i-1)} \right| \\ &= |e^{i-1}| \\ &= e^{-1} < 1. \end{aligned}$$

(In fact, this series is a geometric series with common ratio  $e^{i-1}$ , and  $|e^{i-1}| = e^{-1} < 1$ .)

### Solution to Exercise 2.1

(a) This geometric series converges to  $1/(1-4z)$  when  $|4z| < 1$ , and diverges when  $|4z| > 1$ . Since

$$|4z| < 1 \iff |z| < \frac{1}{4}$$

and

$$|4z| > 1 \iff |z| > \frac{1}{4},$$

the radius of convergence is  $\frac{1}{4}$ .

(b) This geometric series converges to  $1/(1-\alpha z)$  when  $|\alpha z| < 1$ , and diverges when  $|\alpha z| > 1$ . Since

$$|\alpha z| < 1 \iff |z| < 1/|\alpha|$$

and

$$|\alpha z| > 1 \iff |z| > 1/|\alpha|,$$

the radius of convergence is  $1/|\alpha|$ .

## Solution to Exercise 2.2

(a) By the Radius of Convergence Formula (Theorem 2.2), the radius of convergence is

$$\begin{aligned} R &= \lim_{n \rightarrow \infty} \frac{2^n + 4^n}{2^{n+1} + 4^{n+1}} \\ &= \lim_{n \rightarrow \infty} \frac{(1/2)^n + 1}{2(1/2)^n + 4} = \frac{1}{4}. \end{aligned}$$

(b) By the Radius of Convergence Formula, the radius of convergence is

$$\begin{aligned} R &= \lim_{n \rightarrow \infty} \frac{(2n)!/n!}{(2n+2)!/(n+1)!} \\ &= \lim_{n \rightarrow \infty} \frac{n+1}{(2n+2)(2n+1)} \\ &= \lim_{n \rightarrow \infty} \frac{1}{2(2n+1)} = 0. \end{aligned}$$

(c) By the Radius of Convergence Formula, the radius of convergence is

$$\begin{aligned} R &= \lim_{n \rightarrow \infty} \frac{n + 2^{-n}}{(n+1) + 2^{-(n+1)}} \\ &= \lim_{n \rightarrow \infty} \frac{1 + (1/n)2^{-n}}{1 + 1/n + (1/n)2^{-n-1}} = 1. \end{aligned}$$

## Solution to Exercise 2.3

The discs of convergence for the series in Example 2.1 are:

- (a) the empty set  $\emptyset$
- (b) the complex plane  $\mathbb{C}$
- (c) the open disc  $\{z : |z| < 1\}$ .

The discs of convergence for the series in Exercise 2.2 are:

- (a) the open disc  $\{z : |z| < 1/4\}$
- (b) the empty set  $\emptyset$
- (c) the open disc  $\{z : |z - 1| < 1\}$ .

## Solution to Exercise 2.4

(a) By the Radius of Convergence Formula, the radius of convergence is

$$\begin{aligned} R &= \lim_{n \rightarrow \infty} \frac{1/n^2}{1/(n+1)^2} \\ &= \lim_{n \rightarrow \infty} \frac{(n+1)^2}{n^2} \\ &= \lim_{n \rightarrow \infty} (1 + 2/n + 1/n^2) = 1. \end{aligned}$$

(b) By the Radius of Convergence Formula, the radius of convergence is

$$\begin{aligned} R &= \lim_{n \rightarrow \infty} \frac{1/n}{1/(n+1)} \\ &= \lim_{n \rightarrow \infty} \frac{n+1}{n} \\ &= \lim_{n \rightarrow \infty} (1 + 1/n) = 1. \end{aligned}$$

## Solution to Exercise 2.5

We use the fact that the geometric series

$$\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \dots \quad (\text{S1})$$

has disc of convergence  $\{z : |z| < 1\}$  and sum function

$$f(z) = \frac{1}{1-z} \quad (|z| < 1).$$

(a) Since  $\sum_{n=0}^{\infty} (n+1)z^n = \sum_{n=1}^{\infty} nz^{n-1}$ , the power series

$$\sum_{n=0}^{\infty} (n+1)z^n = 1 + 2z + 3z^2 + \dots \quad (\text{S2})$$

can be obtained from the power series (S1) by differentiating term by term.

Hence, by the Differentiation Rule, the power series (S2) has disc of convergence  $\{z : |z| < 1\}$  and sum function

$$g(z) = f'(z) = \frac{1}{(1-z)^2} \quad (|z| < 1).$$

(b) Since

$$\sum_{n=0}^{\infty} (n+1)(n+2)z^n = \sum_{n=1}^{\infty} n(n+1)z^{n-1},$$

the power series

$$\sum_{n=0}^{\infty} (n+1)(n+2)z^n = 2 + 6z + 12z^2 + \dots \quad (\text{S3})$$

can be obtained from the power series (S2) by differentiating term by term.

Hence, by the Differentiation Rule applied to the function

$$g(z) = \frac{1}{(1-z)^2} = \sum_{n=0}^{\infty} (n+1)z^n \quad (|z| < 1)$$

obtained in part (a), we see that the power series (S3) has disc of convergence  $\{z : |z| < 1\}$  and sum function

$$h(z) = g'(z) = \frac{2}{(1-z)^3} \quad (|z| < 1).$$

(c) Since  $\sum_{n=1}^{\infty} \frac{z^n}{n} = \sum_{n=0}^{\infty} \frac{z^{n+1}}{n+1}$ , the power series

$$\sum_{n=1}^{\infty} \frac{z^n}{n} = z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots \quad (\text{S4})$$

can be obtained from the power series (S1) by integrating term by term.

Hence, by the Integration Rule, the function

$$F(z) = b_0 + \sum_{n=1}^{\infty} \frac{z^n}{n},$$

where  $b_0 \in \mathbb{C}$ , is a primitive of  $f$  on the disc of convergence  $\{z : |z| < 1\}$ .

But the function

$$z \mapsto -\text{Log}(1-z)$$

is also a primitive of  $f$  on  $\{z : |z| < 1\}$ . On substituting  $z = 0$  and comparing the two forms of the primitive of  $f$ , we see that  $b_0 = 0$ , so the power series (S4) has sum function

$$z \mapsto -\text{Log}(1-z) \quad (|z| < 1).$$

## Solution to Exercise 2.6

(a) The power series can be written  $\sum_{n=0}^{\infty} (-1)^n z^n$ .

By the Radius of Convergence Formula (Theorem 2.2), the radius of convergence is

$$R = \lim_{n \rightarrow \infty} \left| \frac{(-1)^n}{(-1)^{n+1}} \right| = 1.$$

Hence the disc of convergence is  $\{z : |z| < 1\}$ .

(b) By the Radius of Convergence Formula, the radius of convergence is

$$R = \lim_{n \rightarrow \infty} \left| \frac{(3i)^n}{(3i)^{n+1}} \right| = \frac{1}{3}.$$

Hence the disc of convergence is  $\{z : |z| < \frac{1}{3}\}$ .

(c) The power series can be written as

$$\sum_{n=0}^{\infty} (-1)^n (z - 3i)^n.$$

Similarly to part (a), the radius of convergence is 1.

Hence the disc of convergence is  $\{z : |z - 3i| < 1\}$ .

(d) The power series can be written as

$$\sum_{n=0}^{\infty} 2^n (z - i/2)^n.$$

By the Radius of Convergence Formula, the radius of convergence is

$$R = \lim_{n \rightarrow \infty} \frac{2^n}{2^{n+1}} = \frac{1}{2}.$$

Hence the disc of convergence is  $\{z : |z - i/2| < \frac{1}{2}\}$ .

(e) By the Radius of Convergence Formula, the radius of convergence is

$$R = \lim_{n \rightarrow \infty} \frac{n}{n+1} = 1.$$

Hence the disc of convergence is  $\{z : |z| < 1\}$ .

(f) By the Radius of Convergence Formula, the radius of convergence is

$$R = \lim_{n \rightarrow \infty} \frac{n!}{(n+1)!} = \lim_{n \rightarrow \infty} \frac{1}{n+1} = 0.$$

Hence the disc of convergence is  $\emptyset$ .

(g) By the Radius of Convergence Formula, the radius of convergence is

$$\begin{aligned} R &= \lim_{n \rightarrow \infty} \frac{1/n^n}{1/(n+1)^{n+1}} \\ &= \lim_{n \rightarrow \infty} (n+1) \left( \frac{n+1}{n} \right)^n = \infty, \end{aligned}$$

since  $((n+1)/n)^n > 1$ . Hence the disc of convergence is  $\mathbb{C}$ .

(h) By the Radius of Convergence Formula, the radius of convergence is

$$R = \lim_{n \rightarrow \infty} \frac{1/n!}{1/(n+1)!} = \lim_{n \rightarrow \infty} (n+1) = \infty.$$

Hence the disc of convergence is  $\mathbb{C}$ .

### Solution to Exercise 3.1

Each of the functions  $f$  in this exercise is entire, so its Taylor series about 0 must converge to  $f(z)$  for each  $z \in \mathbb{C}$ .

(a) We have

$$\begin{aligned} f(z) &= \sin z, & \text{so } f(0) &= 0, \\ f^{(1)}(z) &= \cos z, & \text{so } f^{(1)}(0) &= 1, \\ f^{(2)}(z) &= -\sin z, & \text{so } f^{(2)}(0) &= 0, \\ f^{(3)}(z) &= -\cos z, & \text{so } f^{(3)}(0) &= -1, \\ f^{(4)}(z) &= \sin z, & \text{so } f^{(4)}(0) &= 0. \end{aligned}$$

Since every fourth differentiation brings us back to the sine function, the pattern above repeats itself. So the Taylor series about 0 for the function  $f$  is

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots, \quad \text{for } z \in \mathbb{C}.$$

(b) We have

$$\begin{aligned} f(z) &= \cosh z, & \text{so } f(0) &= 1, \\ f^{(1)}(z) &= \sinh z, & \text{so } f^{(1)}(0) &= 0, \\ f^{(2)}(z) &= \cosh z, & \text{so } f^{(2)}(0) &= 1. \end{aligned}$$

Since every second differentiation brings us back to the cosh function, the pattern above repeats itself. So the Taylor series about 0 for the function  $f$  is

$$\cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \cdots, \quad \text{for } z \in \mathbb{C}.$$

(c) We have

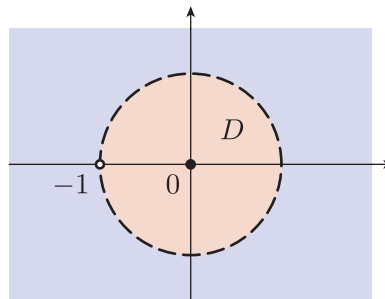
$$\begin{aligned} f(z) &= \sinh z, & \text{so } f(0) &= 0, \\ f^{(1)}(z) &= \cosh z, & \text{so } f^{(1)}(0) &= 1, \\ f^{(2)}(z) &= \sinh z, & \text{so } f^{(2)}(0) &= 0. \end{aligned}$$

Since every second differentiation brings us back to the sinh function, the pattern above repeats itself. So the Taylor series about 0 for the function  $f$  is

$$\sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + \cdots, \quad \text{for } z \in \mathbb{C}.$$

### Solution to Exercise 3.2

The function  $f(z) = (1+z)^{-3}$  is analytic on the region  $\mathbb{C} - \{-1\}$ . The largest open disc centred at 0 that will fit in this region is  $D = \{z : |z| < 1\}$ . So, by Taylor's Theorem, the Taylor series about 0 for  $f$  converges to  $f(z)$  for  $|z| < 1$ .



The Taylor series is found by calculating the higher derivatives of  $f$  at 0. We have

$$\begin{aligned} f(z) &= (1+z)^{-3}, \\ f^{(1)}(z) &= -3(1+z)^{-4}, \\ f^{(2)}(z) &= (-4) \times (-3)(1+z)^{-5}, \\ f^{(3)}(z) &= (-5) \times (-4) \times (-3)(1+z)^{-6}. \end{aligned}$$

It follows that

$$\begin{aligned} f(0) &= 1, \quad f^{(1)}(0) = -3, \quad f^{(2)}(0) = 4 \times 3, \\ f^{(3)}(0) &= -5 \times 4 \times 3, \quad \dots \end{aligned}$$

In general,

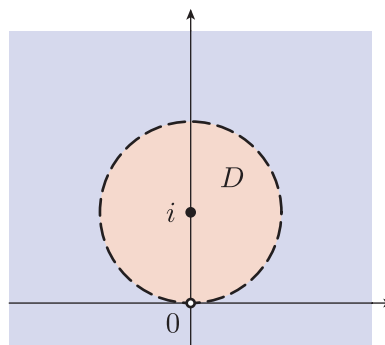
$$\begin{aligned} \frac{f^{(n)}(0)}{n!} &= (-1)^n \frac{(n+2)!}{2 \times n!} \\ &= (-1)^n \frac{(n+2)(n+1)}{2}, \end{aligned}$$

for  $n = 0, 1, 2, \dots$ . The Taylor series about 0 for  $f$  is therefore

$$(1+z)^{-3} = 1 - \frac{3 \times 2}{2}z + \frac{4 \times 3}{2}z^2 - \frac{5 \times 4}{2}z^3 + \cdots, \quad \text{for } |z| < 1.$$

### Solution to Exercise 3.3

The function  $f(z) = 1/z$  is analytic on the region  $\mathbb{C} - \{0\}$ . The largest open disc centred at  $i$  that will fit in this region is  $D = \{z : |z - i| < 1\}$ . So, by Taylor's Theorem, the Taylor series about  $i$  for  $f$  converges to  $f(z)$  for  $|z - i| < 1$ .



The Taylor series is found by calculating the higher derivatives of  $f(z) = 1/z$  at  $i$ . We have

$$\begin{aligned} f(z) &= z^{-1}, & \text{so } f(i) &= -i, \\ f^{(1)}(z) &= -z^{-2}, & \text{so } f^{(1)}(i) &= 1, \\ f^{(2)}(z) &= 2z^{-3}, & \text{so } f^{(2)}(i) &= 2i, \\ f^{(3)}(z) &= -3 \times 2z^{-4}, & \text{so } f^{(3)}(i) &= -3!, \\ f^{(4)}(z) &= 4 \times 3 \times 2z^{-5}, & \text{so } f^{(4)}(i) &= -4!i. \end{aligned}$$

In general,

$$\frac{f^{(n)}(i)}{n!} = \frac{(-1)^n}{i^{n+1}} = \frac{(-1)^n(-i)^n}{i} = i^{n-1},$$

for  $n = 0, 1, 2, \dots$ . The Taylor series about  $i$  for  $f$  is therefore

$$\begin{aligned} \frac{1}{z} &= -i + (z - i) + i(z - i)^2 - \dots \\ &= \sum_{n=0}^{\infty} i^{n-1}(z - i)^n, \quad \text{for } |z - i| < 1. \end{aligned}$$

The general term is  $i^{n-1}(z - i)^n$ .

Alternatively, we can find the required Taylor series by applying a suitable substitution, as follows. First we write

$$f(z) = \frac{1}{z} = \frac{1}{(z - i) + i}.$$

We wish to express this in the form  $a/(1 - w)$ , so that we can use the formula

$$\frac{a}{1 - w} = \sum_{n=0}^{\infty} aw^n, \quad \text{for } |w| < 1,$$

for a geometric series from Theorem 1.2(a). To do this, we multiply the top and bottom of  $1/((z - i) + i)$  by  $-i$  to give

$$f(z) = \frac{-i}{1 - i(z - i)}.$$

It is now of the required form, with  $a = -i$  and  $w = i(z - i)$ . Hence

$$f(z) = -i \sum_{n=0}^{\infty} w^n, \quad \text{for } |w| < 1.$$

That is,

$$\frac{1}{z} = -i \sum_{n=0}^{\infty} i^n (z - i)^n, \quad \text{for } |z - i| < 1,$$

which is equivalent to the answer obtained earlier.

### Solution to Exercise 3.4

Suppose that  $f$  is an odd function. Choose  $w \in f(A)$ . Then  $w = f(z)$ , for some element  $z$  of  $A$ . Since  $f$  is an odd function, we see that

$$f(-z) = -f(z) = -w.$$

Thus  $-w \in f(A)$ . Applying  $f^{-1}$  to each side of the equation  $-w = f(-z)$ , we obtain

$$f^{-1}(-w) = f^{-1}(f(-z)) = -z = -f^{-1}(w),$$

since  $z = f^{-1}(w)$ . Hence  $f^{-1}$  is an odd function.

### Solution to Exercise 3.5

Here the Taylor series is the binomial series with  $\alpha = i$ , that is,

$$(1 + z)^i = 1 + \binom{i}{1}z + \binom{i}{2}z^2 + \binom{i}{3}z^3 + \dots,$$

for  $|z| < 1$ , where

$$\binom{i}{n} = \frac{i(i-1)(i-2)\cdots(i-n+1)}{n!}.$$

The Taylor series is therefore

$$\begin{aligned} 1 + iz + \frac{i(i-1)}{2!}z^2 + \frac{i(i-1)(i-2)}{3!}z^3 + \dots \\ = 1 + iz - \frac{1+i}{2}z^2 + \frac{3+i}{6}z^3 + \dots, \end{aligned}$$

for  $|z| < 1$ .

### Solution to Exercise 3.6

(a) Since the function  $f(z) = \sinh 2z$  is entire, the Taylor series about 0 for  $f$  converges to  $f(z)$  for each  $z \in \mathbb{C}$ . Now

$$\begin{aligned} f(z) &= \sinh 2z, & \text{so } f(0) &= 0, \\ f^{(1)}(z) &= 2 \cosh 2z, & \text{so } f^{(1)}(0) &= 2, \\ f^{(2)}(z) &= 4 \sinh 2z, & \text{so } f^{(2)}(0) &= 0, \\ f^{(3)}(z) &= 8 \cosh 2z, & \text{so } f^{(3)}(0) &= 8, \\ f^{(4)}(z) &= 16 \sinh 2z, & \text{so } f^{(4)}(0) &= 0. \end{aligned}$$

In general,

$$\begin{aligned} f^{(2n)}(0) &= 0, & \text{for } n = 0, 1, 2, \dots, \\ f^{(2n-1)}(0) &= 2^{2n-1}, & \text{for } n = 1, 2, \dots \end{aligned}$$

So  $\sinh 2z$  is equal to

$$2z + \frac{8z^3}{3!} + \frac{32z^5}{5!} + \dots + \frac{2^{2n-1}z^{2n-1}}{(2n-1)!} + \dots,$$

for  $z \in \mathbb{C}$ .

(Here we have followed the usual convention of giving a general term that ignores zero terms.)

Alternatively, we can use the basic Taylor series

$$\sinh w = w + \frac{w^3}{3!} + \frac{w^5}{5!} + \cdots + \frac{w^{2n-1}}{(2n-1)!} + \cdots,$$

for  $w \in \mathbb{C}$ . Substituting  $w = 2z$  gives the Taylor series about 0 for  $f(z) = \sinh 2z$  obtained earlier.

(b) We know that

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots + \frac{(-1)^{n-1} z^{2n-1}}{(2n-1)!} + \cdots,$$

for  $z \in \mathbb{C}$ . Multiplying both sides of this equation by  $z$ , we obtain

$$z \sin z = z^2 - \frac{z^4}{3!} + \frac{z^6}{5!} - \cdots + \frac{(-1)^{n-1} z^{2n}}{(2n-1)!} + \cdots,$$

for  $z \in \mathbb{C}$ .

(c) Since the function  $f(z) = e^{iz}$  is entire, the Taylor series about  $\pi/4$  for  $f$  converges for all  $z \in \mathbb{C}$ . Now

$$\begin{aligned} f(z) &= e^{iz}, & \text{so } f(\pi/4) &= e^{i\pi/4}, \\ f^{(1)}(z) &= ie^{iz}, & \text{so } f^{(1)}(\pi/4) &= ie^{i\pi/4}, \\ f^{(2)}(z) &= i^2 e^{iz}, & \text{so } f^{(2)}(\pi/4) &= i^2 e^{i\pi/4}, \\ f^{(3)}(z) &= i^3 e^{iz}, & \text{so } f^{(3)}(\pi/4) &= i^3 e^{i\pi/4}, \\ f^{(4)}(z) &= i^4 e^{iz}, & \text{so } f^{(4)}(\pi/4) &= i^4 e^{i\pi/4}. \end{aligned}$$

In general,

$$f^{(n)}(\pi/4) = i^n e^{i\pi/4} = i^n (1+i)/\sqrt{2},$$

for  $n = 0, 1, 2, \dots$ . So

$$e^{iz} = \frac{1+i}{\sqrt{2}} \left( 1 + i(z - \pi/4) - \frac{(z - \pi/4)^2}{2!} + \cdots + \frac{i^n (z - \pi/4)^n}{n!} + \cdots \right),$$

for  $z \in \mathbb{C}$ .

Alternatively, observe that

$$f(z) = e^{iz} = e^{i(z-\pi/4)+i\pi/4} = e^{i\pi/4} e^{i(z-\pi/4)}.$$

We know that

$$e^w = 1 + w + \frac{w^2}{2!} + \frac{w^3}{3!} + \cdots + \frac{w^n}{n!} + \cdots,$$

for  $w \in \mathbb{C}$ . Substituting  $w = i(z - \pi/4)$  and then multiplying the resulting power series by

$$e^{i\pi/4} = \frac{1+i}{\sqrt{2}}$$

gives the Taylor series about  $\pi/4$  for  $f(z) = e^{iz}$  obtained earlier.

## Solution to Exercise 4.1

(a) We know that

$$\begin{aligned} \operatorname{Log}(1+z) &= z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots, & \text{for } |z| < 1, \\ (1-z)^{-1} &= 1 + z + z^2 + z^3 + \cdots, & \text{for } |z| < 1. \end{aligned}$$

So, by the Combination Rules,

$$\begin{aligned} h(z) &= 3 + (3+1)z + \left(3 - \frac{1}{2}\right)z^2 + \left(3 + \frac{1}{3}\right)z^3 + \cdots \\ &= 3 + 4z + \frac{5}{2}z^2 + \frac{10}{3}z^3 + \cdots, & \text{for } |z| < 1. \end{aligned}$$

(b) We know that

$$\begin{aligned} \sin z &= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots, & \text{for } z \in \mathbb{C}, \\ \cos z &= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots, & \text{for } z \in \mathbb{C}. \end{aligned}$$

So, by the Combination Rules,

$$h(z) = 1 + z - \frac{z^2}{2!} - \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} - \frac{z^6}{6!} - \frac{z^7}{7!} + \cdots,$$

for  $z \in \mathbb{C}$ .

## Solution to Exercise 4.2

(a) We know that

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots, \quad \text{for } z \in \mathbb{C},$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots, \quad \text{for } z \in \mathbb{C}.$$

Applying the Product Rule, and ignoring powers higher than  $z^3$ , we obtain

$$\begin{aligned} h(z) &= \left(1 + z + \frac{z^2}{2!} + \cdots\right) \left(z - \frac{z^3}{3!} + \cdots\right) \\ &= z + z^2 + \left(-\frac{1}{3!} + \frac{1}{2!}\right)z^3 + \cdots \\ &= z + z^2 + \frac{z^3}{3} + \cdots, & \text{for } z \in \mathbb{C}. \end{aligned}$$

(b) We know that

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots, \quad \text{for } z \in \mathbb{C},$$

$$\operatorname{Log}(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots, \quad \text{for } |z| < 1.$$

Applying the Product Rule, and ignoring powers higher than  $z^3$ , we obtain

$$\begin{aligned} h(z) &= \left(1 - \frac{z^2}{2!} + \cdots\right) \left(z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots\right) \\ &= z - \frac{z^2}{2} + \left(\frac{1}{3} - \frac{1}{2!}\right)z^3 + \cdots \\ &= z - \frac{z^2}{2} - \frac{z^3}{6} + \cdots, \quad \text{for } |z| < 1. \end{aligned}$$

### Solution to Exercise 4.3

(a) Using Theorem 3.3 on binomial series, we have

$$\begin{aligned} (1+w)^{-1/2} &= 1 - \frac{1}{2}w + \frac{1}{2!} \cdot \frac{1}{2} \cdot \frac{3}{2}w^2 \\ &\quad - \frac{1}{3!} \cdot \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2}w^3 + \cdots \\ &= 1 - \frac{1}{2}w + \frac{1 \cdot 3}{2 \cdot 4}w^2 - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}w^3 + \cdots, \end{aligned}$$

for  $|w| < 1$ . Substituting  $w = -z^2$ , we see that

$$(1-z^2)^{-1/2} = 1 + \frac{1}{2}z^2 + \frac{1 \cdot 3}{2 \cdot 4}z^4 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}z^6 + \cdots,$$

for  $|z| < 1$ .

(b) We begin by writing

$$h(z) = \cosh z = \cosh((z - i\pi/2) + i\pi/2).$$

Now let  $w = z - i\pi/2$ . Then

$$\begin{aligned} \cosh z &= \cosh(w + i\pi/2) \\ &= \cosh w \cosh(i\pi/2) + \sinh w \sinh(i\pi/2) \\ &= \cosh w \cos(\pi/2) + i \sinh w \sin(\pi/2) \\ &= i \sinh w. \end{aligned}$$

The Taylor series about 0 for  $\sinh$  is

$$\sinh w = w + \frac{w^3}{3!} + \frac{w^5}{5!} + \cdots, \quad \text{for } w \in \mathbb{C},$$

so

$$\cosh z = i \sinh w = iw + \frac{iw^3}{3!} + \frac{iw^5}{5!} + \cdots,$$

for  $w \in \mathbb{C}$ . By substituting  $w = z - i\pi/2$  into this power series, we see that  $\cosh z$  is equal to

$$i(z - i\pi/2) + \frac{i(z - i\pi/2)^3}{3!} + \frac{i(z - i\pi/2)^5}{5!} + \cdots,$$

for  $z \in \mathbb{C}$ .

(c) We begin by writing

$$h(z) = z^\alpha = (1 + (z - 1))^\alpha,$$

so that  $h(z)$  is expressed in a suitable form to apply Theorem 3.3 on binomial series.

That theorem tells us that

$$(1+w)^\alpha = 1 + \binom{\alpha}{1}w + \binom{\alpha}{2}w^2 + \cdots,$$

for  $|w| < 1$ . Substituting  $w = z - 1$ , we obtain

$$\begin{aligned} z^\alpha &= 1 + \binom{\alpha}{1}(z-1) + \binom{\alpha}{2}(z-1)^2 + \cdots \\ &= 1 + \alpha(z-1) + \frac{\alpha(\alpha-1)}{2}(z-1)^2 + \cdots, \end{aligned}$$

for  $|z-1| < 1$ .

(d) Our strategy is to make a substitution that allows us to find the required Taylor series from the known Taylor series

$$\text{Log}(1+w) = w - \frac{w^2}{2} + \frac{w^3}{3} - \frac{w^4}{4} + \cdots,$$

for  $|w| < 1$ . To do this, we write

$$\begin{aligned} h(z) &= \text{Log}(1+z) = \text{Log}(3 + (z-2)) \\ &= \text{Log}\left(3\left(1 + \frac{1}{3}(z-2)\right)\right). \end{aligned}$$

Now let  $w = \frac{1}{3}(z-2)$ , so  $h(z) = \text{Log}(3(1+w))$ .

If  $|w| < 1$ , then the complex numbers 3 and  $1+w$  both lie in the right half-plane, so we can use the Multiplication Logarithmic Identity

(Theorem 5.1(a) of Unit A2) to see that

$$\begin{aligned} \text{Log}(1+z) &= \text{Log } 3 + \text{Log}(1+w) \\ &= \log 3 + \text{Log}(1+w). \end{aligned}$$

Hence

$$\text{Log}(1+z) = \log 3 + w - \frac{w^2}{2} + \frac{w^3}{3} - \frac{w^4}{4} + \cdots,$$

for  $|w| < 1$ . On substituting  $w = \frac{1}{3}(z-2)$ , we obtain

$$\begin{aligned} \text{Log}(1+z) &= \log 3 + \frac{(z-2)}{3} - \frac{(z-2)^2}{2 \times 3^2} \\ &\quad + \frac{(z-2)^3}{3 \times 3^3} - \frac{(z-2)^4}{4 \times 3^4} + \cdots. \end{aligned}$$

This series converges when  $|\frac{1}{3}(z-2)| < 1$ , that is, when  $|z-2| < 3$ .

### Solution to Exercise 4.4

We know that, for  $z, w \in \mathbb{C}$ ,

$$\exp w = 1 + w + \frac{w^2}{2!} + \frac{w^3}{3!} + \frac{w^4}{4!} + \frac{w^5}{5!} + \cdots,$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots.$$

Let  $w = \sin z$ . Since  $\sin 0 = 0$ , and the Taylor series for  $\sin$  and  $\exp$  are both about 0, we can apply the Composition Rule to obtain the Taylor series about 0 for  $h$ . We see that  $e^{\sin z}$  is equal to

$$\begin{aligned} & 1 + \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots\right) + \frac{1}{2!} \left(z - \frac{z^3}{3!} + \dots\right)^2 \\ & + \frac{1}{3!} \left(z - \frac{z^3}{3!} + \dots\right)^3 + \frac{1}{4!} (z - \dots)^4 \\ & + \frac{1}{5!} (z - \dots)^5 + \dots \\ & = 1 + z + \frac{1}{2!} z^2 + \left(-\frac{1}{3!} + \frac{1}{3!}\right) z^3 \\ & + \left(-\frac{2}{3!2!} + \frac{1}{4!}\right) z^4 + \left(\frac{1}{5!} - \frac{3}{3!3!} + \frac{1}{5!}\right) z^5 + \dots \\ & = 1 + z + \frac{1}{2} z^2 - \frac{1}{8} z^4 - \frac{1}{15} z^5 + \dots, \end{aligned}$$

for  $|z| < r$ , where  $r$  is some positive number.

(In fact, since  $h$  is entire, this representation is valid for all  $z \in \mathbb{C}$ .)

## Solution to Exercise 4.5

First note that

$$\sec z = \frac{1}{\cos z} = \frac{1}{1 - (1 - \cos z)} = g(f(z)).$$

We know that

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots, \quad \text{for } z \in \mathbb{C},$$

so

$$1 - \cos z = \frac{z^2}{2!} - \frac{z^4}{4!} + \frac{z^6}{6!} - \dots, \quad \text{for } z \in \mathbb{C}.$$

The Taylor series about 0 for  $g$  is a geometric series given by

$$\frac{1}{1-w} = 1 + w + w^2 + w^3 + \dots, \quad \text{for } |w| < 1.$$

Let  $w = 1 - \cos z$ . Since  $f(0) = 1 - \cos 0 = 0$ , and the Taylor series for  $f$  and  $g$  are both about 0, we can apply the Composition Rule to obtain the Taylor series about 0 for  $h(z) = g(f(z))$ .

We see that

$$\begin{aligned} \sec z &= 1 + \left(\frac{z^2}{2!} - \frac{z^4}{4!} + \frac{z^6}{6!} - \dots\right) \\ &+ \left(\frac{z^2}{2!} - \frac{z^4}{4!} + \dots\right)^2 + \left(\frac{z^2}{2!} - \dots\right)^3 + \dots \\ &= 1 + \frac{1}{2!} z^2 + \left(-\frac{1}{4!} + \frac{1}{2!2!}\right) z^4 \\ &+ \left(\frac{1}{6!} - \frac{2}{2!4!} + \frac{1}{2!2!2!}\right) z^6 + \dots \\ &= 1 + \frac{1}{2} z^2 + \frac{5}{24} z^4 + \frac{61}{720} z^6 + \dots, \end{aligned}$$

for  $|z| < r$ , where  $r$  is some positive number.

*Remark:* This same Taylor series could have been obtained by applying the Composition Rule to the Taylor series about 0 for  $f(z) = \cos z$  and the Taylor series about 1 for  $g(w) = 1/w$ . This is because, for these functions  $f$  and  $g$ , we have

$$g(f(z)) = 1/\cos z = \sec z,$$

and  $f(0) = \cos 0 = 1$ . The calculations involved are similar, but complicated slightly by the fact that we would need to find the Taylor series for  $g$  about 1 rather than about 0.

## Solution to Exercise 4.6

The domain of the function  $h(z) = \sin^{-1} z$  is

$$\mathbb{C} - \{x : x \in \mathbb{R}, |x| \geq 1\},$$

which contains the unit disc  $\{z : |z| < 1\}$ . And we are given that  $h$  is a primitive of

$$f(z) = (1 - z^2)^{-1/2}$$

(on its domain, and hence on the unit disc).

Furthermore, we know from Exercise 4.3(a) that

$$(1 - z^2)^{-1/2} = 1 + \frac{1}{2} z^2 + \frac{1 \cdot 3}{2 \cdot 4} z^4 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} z^6 + \dots,$$

for  $|z| < 1$ . So, by the Integration Rule, we have

$$\sin^{-1} z = z + \frac{1}{2} \frac{z^3}{3} + \frac{1 \cdot 3}{2 \cdot 4} \frac{z^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \frac{z^7}{7} + \dots,$$

for  $|z| < 1$ , where the constant of integration is zero because  $\sin^{-1} 0 = 0$ .

## Solution to Exercise 4.7

(a) Using Theorem 3.3 on binomial series with  $\alpha = \frac{1}{2}$ , we obtain

$$\begin{aligned}(1+z)^{1/2} &= 1 + \frac{1}{2}z - \frac{1}{2!} \cdot \frac{1}{2} \cdot \frac{1}{2} z^2 \\ &\quad + \frac{1}{3!} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{3}{2} z^3 + \cdots \\ &= 1 + \frac{1}{2}z - \frac{1}{8}z^2 + \frac{1}{16}z^3 - \cdots,\end{aligned}$$

for  $|z| < 1$ .

(b) Using the result of part (a) twice, once as given and once with  $z$  replaced by  $-z$ , we see from the Combination Rules that

$$\begin{aligned}(1+z)^{1/2} - (1-z)^{1/2} &= 1 + \frac{1}{2}z - \frac{1}{8}z^2 + \frac{1}{16}z^3 - \cdots \\ &\quad - \left(1 - \frac{1}{2}z - \frac{1}{8}z^2 - \frac{1}{16}z^3 - \cdots\right) \\ &= z + \frac{1}{8}z^3 + \cdots,\end{aligned}$$

for  $|z| < 1$ .

(c) For  $|z| < 1$ , we see that  $1-z$ ,  $1+z$  and  $(1+z)^{-1}$  lie in the right half-plane  $\{z : \operatorname{Re} z > 0\}$ . Hence, by the Logarithmic Identities (Theorem 5.1 of Unit A2),

$$\begin{aligned}\operatorname{Log}\left(\frac{1-z}{1+z}\right) &= \operatorname{Log}(1-z) + \operatorname{Log}\left(\frac{1}{1+z}\right) \\ &= \operatorname{Log}(1-z) - \operatorname{Log}(1+z) \\ &= \left(-z - \frac{z^2}{2} - \frac{z^3}{3} - \cdots\right) \\ &\quad - \left(z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots\right) \\ &= -2z - \frac{2}{3}z^3 - \cdots,\end{aligned}$$

for  $|z| < 1$ .

(d) On replacing  $z$  by  $z^2$  in the Taylor series for  $\cos$ , we obtain

$$\begin{aligned}z^3 \cos(z^2) &= z^3 \left(1 - \frac{(z^2)^2}{2!} + \frac{(z^2)^4}{4!} - \cdots\right), \\ &= z^3 - \frac{z^7}{2!} + \frac{z^{11}}{4!} - \cdots,\end{aligned}$$

for  $z \in \mathbb{C}$ .

(e) By the Product Rule,

$$\begin{aligned}(\sin z)(\cos z) &= \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right) \\ &\quad \times \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots\right) \\ &= z + \left(-\frac{1}{3!} - \frac{1}{2!}\right)z^3 \\ &\quad + \left(\frac{1}{5!} + \frac{1}{2!3!} + \frac{1}{4!}\right)z^5 + \cdots \\ &= z - \frac{2}{3}z^3 + \frac{2}{15}z^5 - \cdots,\end{aligned}$$

for  $z \in \mathbb{C}$ .

Alternatively,

$$\begin{aligned}(\sin z)(\cos z) &= \frac{1}{2} \sin 2z \\ &= \frac{1}{2} \left(2z - \frac{(2z)^3}{3!} + \frac{(2z)^5}{5!} - \cdots\right) \\ &= z - \frac{2}{3}z^3 + \frac{2}{15}z^5 - \cdots,\end{aligned}$$

for  $z \in \mathbb{C}$ .

(f) We have

$$\begin{aligned}\cosh z &= 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \cdots, \quad \text{for } z \in \mathbb{C}, \\ \operatorname{Log} w &= (w-1) - \frac{(w-1)^2}{2} + \frac{(w-1)^3}{3} - \cdots, \\ &\quad \text{for } |w-1| < 1,\end{aligned}$$

by Example 3.4. Since  $\cosh 0 = 1$ , we can apply the Composition Rule to find  $r > 0$  such that

$$\begin{aligned}\operatorname{Log}(\cosh z) &= \operatorname{Log}\left(1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots\right) \\ &= \left(\frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \cdots\right) \\ &\quad - \frac{1}{2} \left(\frac{z^2}{2!} + \frac{z^4}{4!} + \cdots\right)^2 \\ &\quad + \frac{1}{3} \left(\frac{z^2}{2!} + \frac{z^4}{4!} + \cdots\right)^3 - \cdots \\ &= \frac{1}{2}z^2 + \left(\frac{1}{4!} - \frac{1}{2(2!)^2}\right)z^4 \\ &\quad + \left(\frac{1}{6!} - \frac{1}{2!4!} + \frac{1}{3(2!)^3}\right)z^6 + \cdots \\ &= \frac{1}{2}z^2 - \frac{1}{12}z^4 + \frac{1}{45}z^6 - \cdots,\end{aligned}$$

for  $|z| < r$ .

Alternatively, substitute  $iz$  for  $z$  in the Taylor series about 0 for  $\text{Log}(\cos z)$  obtained in Example 4.6.

(g) Since  $\tanh z = g'(z)$ , for  $|z| < r$ , where  $g(z) = \text{Log}(\cosh z)$  and  $r$  is the same as in part (f), it follows, from the Differentiation Rule applied to the series in part (f), that the Taylor series about 0 for  $\tanh$  is

$$\tanh z = z - \frac{1}{3}z^3 + \frac{2}{15}z^5 - \dots, \quad \text{for } |z| < r.$$

*Remark:* The domain of  $\tanh$  is  $\mathbb{C} - \{(n + \frac{1}{2})\pi i : n \in \mathbb{Z}\}$ , as we saw in Subsection 4.3 of Unit A2. The function  $\tanh$  is analytic on its domain, so it is analytic on the open disc  $\{z : |z| < \pi/2\}$ . But it is unbounded on this disc, because it ‘blows up’ at the point  $i\pi/2$ . Hence the radius of convergence of the Taylor series about 0 for  $\tanh$  is  $\pi/2$ , by Theorem 4.4.

## Solution to Exercise 5.1

(a) Since  $f(z) = z^3(z-1)^4(z+2) = 0$  when  $z$  is 0, 1 or  $-2$ , these are the three zeros of  $f$ . Now

$$f(z) = z^3g(z),$$

where  $g(z) = (z-1)^4(z+2)$  is analytic and non-zero at 0. So  $f$  has a zero of order 3 at 0. Similarly,  $f$  has a zero of order 4 at 1, and a simple zero at  $-2$ .

(b) Here  $f$  has only a single zero, at 3. Now

$$f(z) = (z-3)g(z),$$

where  $g(z) = 1/(z+2)$  is analytic and non-zero at 3. So  $f$  has a simple zero at 3.

(c) Observe that

$$f(z) = \frac{(z^2+9)^3}{(z^2+4)e^z} = \frac{(z-3i)^3(z+3i)^3}{(z^2+4)e^z}.$$

Since  $f(z) = 0$  when  $z$  is  $3i$  or  $-3i$ , these are the two zeros of  $f$ . Now

$$f(z) = (z-3i)^3g(z),$$

where  $g(z) = (z+3i)^3e^{-z}/(z^2+4)$  is analytic and non-zero at  $3i$ . So  $f$  has a zero of order 3 at  $3i$ . Similarly,  $f$  has a zero of order 3 at  $-3i$ .

## Solution to Exercise 5.2

(a) Although we can factor out  $z^4$ , that still leaves  $\sin 2z$ , which is zero at 0. So, using the Taylor series about 0 for  $\sin$ , we obtain

$$\begin{aligned} f(z) &= z^4 \sin 2z \\ &= z^4 \left( 2z - \frac{(2z)^3}{3!} + \frac{(2z)^5}{5!} - \dots \right) \\ &= z^5 \left( 2 - \frac{2^3}{3!}z^2 + \frac{2^5}{5!}z^4 - \dots \right) \\ &= z^5 g(z), \quad \text{for } z \in \mathbb{C}, \end{aligned}$$

where

$$g(z) = 2 - \frac{2^3}{3!}z^2 + \frac{2^5}{5!}z^4 - \dots,$$

which is analytic and non-zero at 0. Thus  $f$  has a zero of order 5 at 0.

(b) Although we can factor out  $z^2$ , that still leaves  $\cos z - 1$ , which is zero at 0. So, using the Taylor series about 0 for  $\cos$ , we obtain

$$\begin{aligned} f(z) &= z^2(\cos z - 1) \\ &= z^2 \left( -\frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots \right) \\ &= z^4 \left( -\frac{1}{2!} + \frac{z^2}{4!} - \frac{z^4}{6!} + \dots \right) \\ &= z^4 g(z), \quad \text{for } z \in \mathbb{C}, \end{aligned}$$

where

$$g(z) = -\frac{1}{2!} + \frac{z^2}{4!} - \frac{z^4}{6!} + \dots,$$

which is analytic and non-zero at 0. Thus  $f$  has a zero of order 4 at 0.

(c) To find the order of the zero at 0, we use the Taylor series about 0 for  $\sin$  to obtain

$$\begin{aligned} f(z) &= 6 \sin(z^2) + z^2(z^4 - 6) \\ &= 6 \left( z^2 - \frac{z^6}{3!} + \frac{z^{10}}{5!} - \dots \right) + (z^6 - 6z^2) \\ &= 6 \left( \frac{z^{10}}{5!} - \frac{z^{14}}{7!} + \dots \right) \\ &= z^{10} \left( \frac{6}{5!} - \frac{6}{7!}z^4 + \dots \right) \\ &= z^{10} g(z), \quad \text{for } z \in \mathbb{C}, \end{aligned}$$

where

$$g(z) = \frac{6}{5!} - \frac{6}{7!}z^4 + \dots,$$

which is analytic and non-zero at 0. Thus  $f$  has a zero of order 10 at 0.

### Solution to Exercise 5.3

Since  $f$  and  $g$  are represented by the same Taylor series on  $D$ , it follows that  $f(z) = g(z)$  for all  $z \in D$ . Also,  $D$  has a limit point in  $\mathcal{R}$ ; indeed every point of  $D$  is such a point. The conditions of the Uniqueness Theorem are therefore satisfied, so  $f$  agrees with  $g$  on  $\mathcal{R}$ .

### Solution to Exercise 5.4

By the Uniqueness Theorem,  $f = g$  provided that  $f$  and  $g$  agree on a set  $S$  with a limit point (in  $\mathbb{C}$ ).

(a) This condition is sufficient since  $S$  has a limit point (for example, 2).

(b) Here  $S = \mathbb{N}$  has no limit points, so the condition may not be sufficient. In fact, the analytic functions  $f(z) = \sin \pi z$  and  $g(z) = 0$  agree on  $\mathbb{N}$  but are not equal, so the condition is not sufficient.

(c) In this case  $S$  has limit point 0, so the condition is sufficient.

### Solution to Exercise 5.5

Let  $g(z) = z^2$ . Then the entire functions  $f$  and  $g$  agree on the set  $S = \{i/n : n = 1, 2, \dots\}$  because, for  $n = 1, 2, \dots$ , we have

$$g(i/n) = (i/n)^2 = -1/n^2 = f(i/n).$$

Since  $S$  has limit point  $0 \in \mathbb{C}$ , we deduce, by the Uniqueness Theorem, that  $f$  agrees with  $g$  throughout  $\mathbb{C}$ . Hence  $f(z) = z^2$ , for  $z \in \mathbb{C}$ .

### Solution to Exercise 5.6

(a) The function  $f(z) = z^2(z^2 + 4)^3$  can be written as

$$f(z) = z^2(z + 2i)^3(z - 2i)^3.$$

So  $f$  has a zero of order 2 at 0, and zeros of order 3 at  $-2i$  and  $2i$ .

(b) The function  $f(z) = z \sin z$  has zeros at  $k\pi$ , for  $k \in \mathbb{Z}$ .

The Taylor series about 0 for  $f$  is

$$\begin{aligned} f(z) &= z \left( z - \frac{z^3}{3!} + \dots \right) \\ &= z^2 \left( 1 - \frac{z^2}{3!} + \dots \right), \quad \text{for } z \in \mathbb{C}. \end{aligned}$$

So  $f$  has a zero of order 2 at 0.

The Taylor series about  $k\pi$  for  $f$ , where  $k \in \mathbb{Z} - \{0\}$ , is

$$\begin{aligned} f(z) &= z \sin z \\ &= z(-1)^k \sin(z - k\pi) \\ &= z(-1)^k \left( (z - k\pi) - \frac{(z - k\pi)^3}{3!} + \dots \right) \\ &= (z - k\pi)z(-1)^k \left( 1 - \frac{(z - k\pi)^2}{3!} + \dots \right), \end{aligned}$$

for  $z \in \mathbb{C}$ . So  $f$  has a zero of order 1 at  $k\pi$ , for  $k \in \mathbb{Z} - \{0\}$ .

Unit B4

Laurent series



# Introduction

In Unit B3 you learned that a function  $f$  that is analytic at a point  $\alpha$  can be represented by a Taylor series

$$f(z) = a_0 + a_1(z - \alpha) + a_2(z - \alpha)^2 + \cdots, \quad \text{for } |z - \alpha| < r,$$

where  $r$  is some positive number. For example, if  $f(z) = \sin z$  and  $\alpha = 0$ , then

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots, \quad \text{for } z \in \mathbb{C}.$$

If, however, the function  $f$  is *not* analytic at the point  $\alpha$ , then it is not possible to represent  $f$  by a Taylor series about  $\alpha$ . Nonetheless, as we will see, provided that  $f$  is analytic on some *punctured* open disc

$$\{z : 0 < |z - \alpha| < r\},$$

as illustrated in Figure 0.1, it is still possible to represent  $f$  by a series, although we will need to allow negative powers of  $z$ . For example, if

$$f(z) = \frac{\sin z}{z^3} \quad \text{and} \quad \alpha = 0,$$

then  $f$  is not analytic at 0, but it is analytic on any punctured open disc with centre 0, and can be represented by the series

$$\begin{aligned} \frac{\sin z}{z^3} &= \frac{1}{z^3} \left( z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots \right) \\ &= \frac{1}{z^2} - \frac{1}{3!} + \frac{z^2}{5!} - \frac{z^4}{7!} + \cdots, \quad \text{for } z \in \mathbb{C} - \{0\}. \end{aligned}$$

We say that this function  $f$  has an *isolated singularity* at 0.

An extended power series like this series for  $(\sin z)/z^3$ , with negative powers of  $z$ , is called a *Laurent series*. Sometimes we need to allow infinitely many negative powers of  $z$ . For example,

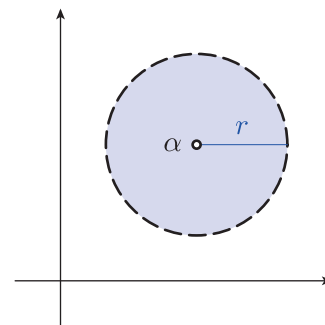
$$\begin{aligned} \sin\left(\frac{1}{z}\right) &= \left(\frac{1}{z}\right) - \frac{1}{3!}\left(\frac{1}{z}\right)^3 + \frac{1}{5!}\left(\frac{1}{z}\right)^5 - \cdots \\ &= \frac{1}{z} - \frac{1}{3!z^3} + \frac{1}{5!z^5} - \cdots, \quad \text{for } z \in \mathbb{C} - \{0\}. \end{aligned}$$

This unit is about isolated singularities and Laurent series, and how they relate to each other.

In Section 1 we discuss isolated singularities and describe how to classify them into three types: removable singularities, poles and essential singularities.

In Section 2 we state Laurent's Theorem, which asserts that any function that is analytic on a punctured open disc (or on an open annulus) can be represented by a Laurent series, and that this representation is unique. We then relate Laurent series to singularities.

In Section 3 we investigate the behaviour of a given function near a singularity. This behaviour turns out to be very different for the three types of singularity.



**Figure 0.1** A punctured open disc

Finally, in Section 4 we focus our attention on just one of the coefficients of the Laurent series, called the *residue*, and we show how residues can be used to help us to evaluate integrals. (This is a theme that we will return to in Unit C1.) We conclude with a short revision subsection, designed to test your understanding of many of the results in Book B.

## Unit guide

Sections 1 and 2 are both lengthy, but contain ideas that are crucial to your understanding of the rest of the module – namely, classifying singularities and calculating Laurent series. You should try to ensure that you understand the material in these two sections before proceeding further; however, if you are short of time, then you can omit Subsection 2.3 (which contains a proof of Laurent’s Theorem) on a first reading.

Section 3 is largely theoretical. You may choose to read the statements of the theorems but omit the proofs until later.

The material in Subsection 4.1 is crucial to Book C and will be reviewed at the beginning of Unit C1. The revision exercise in Subsection 4.2 should help you to consolidate the techniques from Book B.

# 1 Singularities

After working through this section, you should be able to:

- explain the term (*isolated*) *singularity*
- explain what is meant by the phrase ‘ $f(z) \rightarrow \infty$  as  $z \rightarrow \alpha$ ’
- classify a given singularity as a *removable singularity*, a *pole* or an *essential singularity*
- describe the behaviour of a given function near a removable singularity or a pole.

## 1.1 Examples of singularities

Consider the following three functions:

$$f_1(z) = \frac{z+i}{z^2(z-2)}, \quad f_2(z) = \frac{\cos z}{\sin z}, \quad f_3(z) = z \sin\left(\frac{1}{z}\right).$$

Each function is not analytic at 0, but is analytic at all points near 0; that is, each function is analytic on a punctured open disc with centre 0. For example:

$f_1$  is not analytic at 0, but is analytic on the punctured open disc  $\{z : 0 < |z| < 2\}$

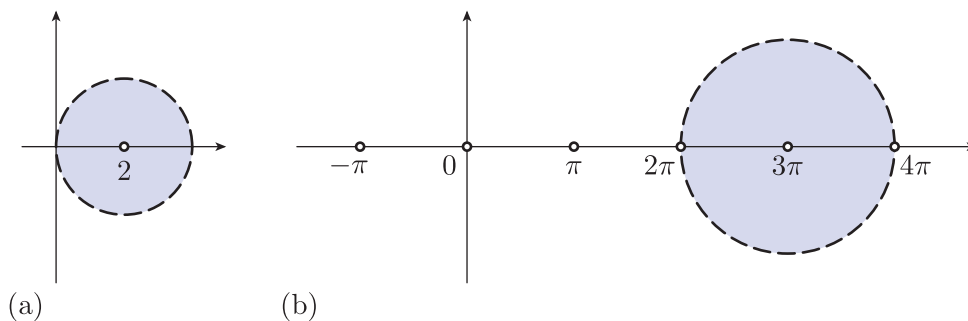
$f_2$  is not analytic at 0, but is analytic on the punctured open disc  $\{z : 0 < |z| < \pi\}$

$f_3$  is not analytic at 0, but is analytic on  $\mathbb{C} - \{0\}$ , so it is analytic on any punctured open disc centred at 0.

We say that each of these functions has an *isolated singularity* at 0.

Note that the function  $f_1$  also has a singularity at the point 2, since  $f_1$  is not analytic at 2 but is analytic at all points near 2; in particular, it is analytic on the punctured open disc  $\{z : 0 < |z - 2| < 2\}$  (see Figure 1.1(a)).

Similarly, the function  $f_2$  has singularities at all points where  $\sin z = 0$ , namely  $0, \pm\pi, \pm2\pi, \dots$ . For example,  $f_2$  is not analytic at  $3\pi$  but is analytic at all points near  $3\pi$ ; in particular, it is analytic on the punctured open disc  $\{z : 0 < |z - 3\pi| < \pi\}$  (see Figure 1.1(b)).



**Figure 1.1** The punctured open discs (a)  $\{z : 0 < |z - 2| < 2\}$  and (b)  $\{z : 0 < |z - 3\pi| < \pi\}$

We can now give the general definition of an isolated singularity.

### Definition

A function  $f$  has an **isolated singularity** at the point  $\alpha$  if  $f$  is analytic on a punctured open disc  $\{z : 0 < |z - \alpha| < r\}$ , where  $r > 0$ , but not at  $\alpha$  itself.

### Remarks

1. Note that the function  $f$  may or may not be defined at the point  $\alpha$ . If  $f$  is defined at  $\alpha$ , then it is not analytic there; for example, the function

$$f(z) = \begin{cases} z, & z \neq 0, \\ 2, & z = 0, \end{cases}$$

has an isolated singularity at 0.

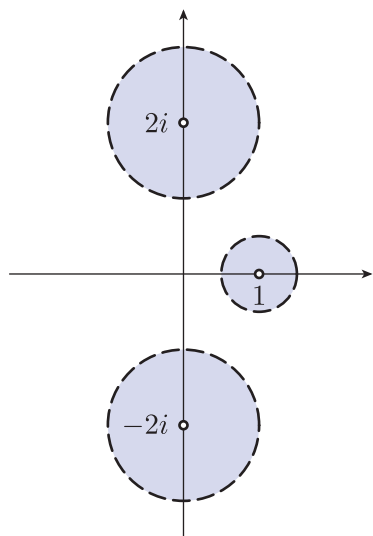
2. We usually drop the word ‘isolated’ and say, simply, that  $f$  has a *singularity* at  $\alpha$ . Alternatively, we often say ‘ $\alpha$  is a singularity of  $f$ ’.

### Example 1.1

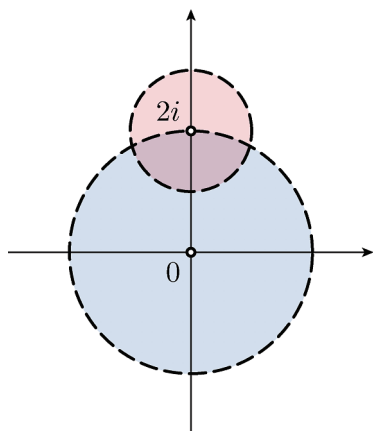
Locate all the singularities of each of the following functions.

$$(a) \ f(z) = \frac{3z}{(z-1)(z^2+4)} \quad (b) \ f(z) = \frac{e^{1/z}}{(z-2i)^2}$$

$$(c) \ f(z) = \frac{z+1}{\cos 2z}$$



**Figure 1.2** Punctured open discs centred at 1,  $2i$  and  $-2i$



**Figure 1.3** Punctured open discs centred at 0 and  $2i$

### Solution

(a) The function

$$f(z) = \frac{3z}{(z-1)(z^2+4)}$$

is analytic everywhere except at the zeros of the denominator. Thus the only possible singularities are at 1,  $2i$  and  $-2i$ . Clearly, about each of these points we can find a punctured open disc on which  $f$  is analytic – for example:

about 1, the punctured disc  $\{z : 0 < |z-1| < \frac{1}{2}\}$

about  $2i$ , the punctured disc  $\{z : 0 < |z-2i| < 1\}$

about  $-2i$ , the punctured disc  $\{z : 0 < |z+2i| < 1\}$

(see Figure 1.2). Hence the singularities of  $f$  are at 1,  $2i$ ,  $-2i$ .

(b) The function

$$f(z) = \frac{e^{1/z}}{(z-2i)^2}$$

is analytic everywhere except at 0, where the numerator is undefined, and at  $2i$ , where the denominator is zero. Thus the only possible singularities are at 0 and  $2i$ . Clearly, about each of these points we can find a punctured open disc on which  $f$  is analytic – for example:

about 0, the punctured disc  $\{z : 0 < |z| < 2\}$

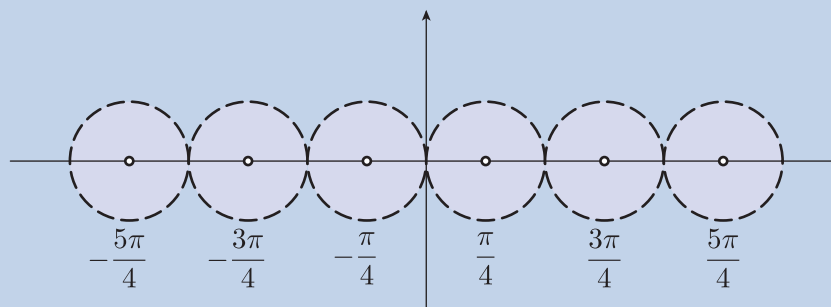
about  $2i$ , the punctured disc  $\{z : 0 < |z-2i| < 1\}$

(see Figure 1.3). Hence the singularities of  $f$  are at 0,  $2i$ .

(c) The function

$$f(z) = \frac{z+1}{\cos 2z}$$

is analytic everywhere except when  $\cos 2z = 0$ , that is, when  $2z = \pm\pi/2, \pm3\pi/2, \pm5\pi/2, \dots$ . Thus the only possible singularities are at  $\pm\pi/4, \pm3\pi/4, \pm5\pi/4, \dots$ . Clearly, about each of these points we can find a punctured open disc (of radius  $\pi/4$ , say) on which  $f$  is analytic (see Figure 1.4).



**Figure 1.4** Punctured open discs centred at  $\pm\pi/4, \pm3\pi/4, \pm5\pi/4$

Hence the singularities of  $f$  are at  $\pm\pi/4, \pm3\pi/4, \pm5\pi/4, \dots$

Note that in Example 1.1 we did not always choose the largest possible punctured open disc; any convenient choice will do.

### Exercise 1.1

Locate the singularities of each of the following functions.

$$(a) f(z) = \frac{z + 2i}{(z - 3)^2(z^2 + 1)} \quad (b) f(z) = \frac{3z - i}{z^3} \sin\left(\frac{1}{z + 1}\right)$$

$$(c) f(z) = \frac{4e^{-z}}{z^2 + 2iz - 1}$$

Great care must be taken when identifying singularities. Consider the function

$$f(z) = \frac{1}{\sqrt{z}}.$$

At first sight, it might seem that  $f$  has a singularity at 0. However, remember that we are using ‘singularity’ to mean ‘isolated singularity’, and the function  $z \mapsto \sqrt{z}$  fails to be analytic at each point of the negative real axis (see Example 1.4 of Unit A4). Thus there is no punctured open disc centred at 0 on which  $f$  is analytic, so  $f$  does not have a singularity at 0.

### Exercise 1.2

Consider the function

$$f(z) = \frac{1}{\sin(1/z)}.$$

Show that  $f$  has singularities at the points

$$\pm\frac{1}{\pi}, \pm\frac{1}{2\pi}, \pm\frac{1}{3\pi}, \dots$$

Explain why 0 is not a singularity of  $f$ .

## 1.2 Functions tending to infinity

In the next subsection we will describe the various types of singularity. To do this, we need the concept of a complex function tending to infinity.

Consider the function

$$f(z) = \frac{1}{z^2},$$

which has a singularity at 0. If  $(z_n)$  is any sequence of non-zero complex numbers that tends to 0, then  $z_n^2 \rightarrow 0$  as  $n \rightarrow \infty$ . Hence, by the Reciprocal Rule,

$$\frac{1}{z_n^2} \rightarrow \infty \text{ as } n \rightarrow \infty;$$

that is,

$$f(z_n) \rightarrow \infty \text{ as } n \rightarrow \infty.$$

This behaviour of  $f$  can be expressed as

$$f(z) \rightarrow \infty \text{ as } z \rightarrow 0.$$

More generally, we have the following definition.

### Definition

Let  $f$  be a function with domain  $A$ , and suppose that  $\alpha$  is a limit point of  $A$ .

The function  $f$  **tends to infinity as  $z$  tends to  $\alpha$**  if, for each sequence  $(z_n)$  in  $A - \{\alpha\}$  such that  $z_n \rightarrow \alpha$ , we have  $f(z_n) \rightarrow \infty$ .

We write

$$f(z) \rightarrow \infty \text{ as } z \rightarrow \alpha.$$

Observe that we usually avoid writing

$$\lim_{z \rightarrow \alpha} f(z) = \infty,$$

because this might suggest that  $\infty$  is a complex number, although you may see this notation used in other texts.

The related concept

$$f(z) \rightarrow \beta \text{ as } z \rightarrow \alpha,$$

where  $\alpha, \beta \in \mathbb{C}$ , was introduced in Section 3 of Unit A3. There we gave an alternative definition of the limiting process using  $\varepsilon$ - $\delta$  notation. In a similar way, the limiting process

$$f(z) \rightarrow \infty \text{ as } z \rightarrow \alpha,$$

which we have defined using sequences, has the following equivalent definition.

### Definition

Let  $f$  be a function with domain  $A$ , and suppose that  $\alpha$  is a limit point of  $A$ .

The function  $f$  **tends to infinity as  $z$  tends to  $\alpha$**  if, for each  $M > 0$ , there is a  $\delta > 0$  such that

$$|f(z)| > M, \quad \text{for all } z \in A \text{ with } 0 < |z - \alpha| < \delta.$$

As with sequences, there is a version of the Reciprocal Rule that relates the behaviour of functions that tend to infinity to that of functions that tend to 0. You are asked to prove the rule in Exercise 1.4.

**Theorem 1.1 Reciprocal Rule for Functions**

Let  $f$  be a function with domain  $A$ , and suppose that  $\alpha$  is a limit point of  $A$ . Then

$$f(z) \rightarrow \infty \text{ as } z \rightarrow \alpha$$

if and only if

$$\lim_{z \rightarrow \alpha} \frac{1}{f(z)} = 0.$$

For example, taking  $f(z) = 1/z^2$ , we have

$$f(z) = \frac{1}{z^2} \rightarrow \infty \text{ as } z \rightarrow 0$$

because 0 is a limit point of the domain  $\mathbb{C} - \{0\}$  of  $f$ , and

$$\lim_{z \rightarrow 0} \frac{1}{f(z)} = \lim_{z \rightarrow 0} z^2 = 0.$$

The next exercise gives you practice at applying the Reciprocal Rule. For part (b), you will find it helpful to use the following limit.

$$\lim_{z \rightarrow 0} \frac{\sin z}{z} = 1.$$

This result can be proved by using the fact that the derivative of  $\sin$  is  $\cos$  (established in Subsection 3.1 of Unit A4), so

$$\lim_{z \rightarrow 0} \frac{\sin z}{z} = \lim_{z \rightarrow 0} \frac{\sin z - \sin 0}{z - 0} = \cos 0 = 1.$$

This limit will be used many times in this unit.

**Exercise 1.3**

Use the Reciprocal Rule to prove that

- (a)  $\frac{1}{z-i} \rightarrow \infty$  as  $z \rightarrow i$
- (b)  $\frac{\sin z}{z^2} \rightarrow \infty$  as  $z \rightarrow 0$
- (c)  $\frac{z+i}{(z^2+1)^3} \rightarrow \infty$  as  $z \rightarrow i$ .

## Exercise 1.4

Prove the Reciprocal Rule for functions.

You may assume the Reciprocal Rule for sequences (Theorem 1.5 of Unit A3).

## 1.3 Classifying singularities

In this subsection we will classify singularities into three types: *removable singularities*, *poles* and *essential singularities*. Of these, poles will be the most important for our later work, and we will see how to classify them further as poles of order 1 (*simple poles*), poles of order 2, and so on.

We also state some observations about the behaviour of a function near a removable singularity and near a pole. Proofs of these observations will be given in Section 3, as parts of the proofs of more general results.

To begin our classification, let us consider the function

$$f(z) = \frac{\sin z}{z},$$

which is analytic on  $\mathbb{C} - \{0\}$ . However, it is not defined at 0, so it has a singularity at 0. The Taylor series about 0 for  $\sin$  is

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots, \quad \text{for } z \in \mathbb{C}.$$

If  $z \neq 0$ , then we can divide both sides of this equation by  $z$  to obtain

$$\frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots, \quad \text{for } z \in \mathbb{C} - \{0\}.$$

This power series also converges when  $z = 0$  (to the number 1), so we can define a new function  $g$  by

$$g(z) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots, \quad \text{for } z \in \mathbb{C}.$$

The new function  $g$  is equal to  $f$  on  $\mathbb{C} - \{0\}$ , but, unlike  $f$ , it is also defined at  $z = 0$ , with value  $g(0) = 1$ . What is more,  $g$  is analytic at 0 because it is defined by a convergent power series.

Observe that the value of  $g$  at 0 can be written in terms of  $f$  as

$$\lim_{z \rightarrow 0} f(z) = \lim_{z \rightarrow 0} g(z) = g(0) = 1,$$

where we have used the property that  $g$  is analytic, and hence continuous, at 0. In fact,  $\lim_{z \rightarrow 0} f(z) = 1$  is just another way of writing the limit

$$\lim_{z \rightarrow 0} \frac{\sin z}{z} = 1,$$

which we obtained in the previous subsection by using the derivative of  $\sin$  at 0.

In a sense we have ‘removed’ the singularity of  $f$  at 0 by extending the definition of  $f$  to a function that has value  $\lim_{z \rightarrow 0} f(z)$  at 0. For this reason,  $f$  is said to have a *removable singularity* at 0.

More generally, we make the following definitions.

### Definitions

Let  $f$  be a function that has a singularity at the point  $\alpha$ .

Then  $f$  has a **removable singularity** at  $\alpha$  if there is a function  $g$  that is analytic on an open disc  $\{z : |z - \alpha| < r\}$  such that

$$f(z) = g(z), \quad \text{for } 0 < |z - \alpha| < r.$$

The function  $g$  is called an **analytic extension** of  $f$  to  $\{z : |z - \alpha| < r\}$ .

Notice that in the circumstances described by the definition of a removable singularity, we have

$$\lim_{z \rightarrow \alpha} f(z) = \lim_{z \rightarrow \alpha} g(z) = g(\alpha),$$

since  $g$  is analytic, and hence continuous, at  $\alpha$ . We highlight this observation below, for later use. In fact, it is one part of a more general result, Theorem 3.1, which we will meet in Section 3.

### Observation

Let  $f$  be a function that has a removable singularity at the point  $\alpha$ .

Then  $\lim_{z \rightarrow \alpha} f(z)$  exists.

A consequence of this observation is that if  $\lim_{z \rightarrow \alpha} f(z)$  does *not* exist, then  $f$  does *not* have a removable singularity at  $\alpha$ .

Later on we will often denote an analytic extension by the same letter as the original function, writing  $f$  for both functions rather than using  $g$  and  $f$ . For the time being, however, we will distinguish the two functions by different letters.

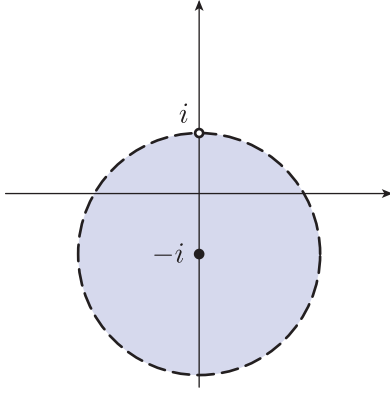
### Example 1.2

Locate the removable singularities of the function

$$f(z) = \frac{z+i}{z^2+1}.$$

### Solution

The function  $f$  has singularities at  $-i$  and  $i$  (when  $z^2 + 1 = 0$ ) and it is analytic on  $\mathbb{C} - \{-i, i\}$ .



**Figure 1.5** The disc  $\{z : |z + i| < 2\}$

Let us consider the singularity at  $-i$  first. Since

$$z^2 + 1 = (z + i)(z - i),$$

we can write

$$f(z) = \frac{z + i}{z^2 + 1} = \frac{z + i}{(z + i)(z - i)} = \frac{1}{z - i}, \quad \text{for } z \in \mathbb{C} - \{-i, i\}.$$

Now, we can define an analytic function

$$g(z) = \frac{1}{z - i} \quad (z \in \mathbb{C} - \{i\}).$$

This function is analytic on the disc

$$\{z : |z + i| < 2\} = \{z : |z - (-i)| < 2\}$$

illustrated in Figure 1.5. Furthermore,

$$f(z) = g(z), \quad \text{for } 0 < |z + i| < 2,$$

so  $g$  is an analytic extension of  $f$  to  $\{z : |z + i| < 2\}$ . Hence  $f$  has a removable singularity at  $-i$ .

Next, observe that

$$f(z) = \frac{1}{z - i} \rightarrow \infty \text{ as } z \rightarrow i,$$

by Exercise 1.3(a). Hence  $f$  does not have a removable singularity at  $i$ .

### Exercise 1.5

Locate the removable singularities of each of the following functions.

$$(a) \ f(z) = \frac{\sin^2 z}{z^2} \quad (b) \ f(z) = 3z \cot z \quad (c) \ f(z) = \frac{z^2 + 3iz - 2}{z^2 + 4}$$

(Hint: For part (b), consider the singularity  $z = 0$  separately from other singularities.)

Let us move on to consider another type of singularity, which we explain using the function

$$f(z) = \frac{\sin z}{z^2} \quad (z \in \mathbb{C} - \{0\}).$$

This function has a singularity at 0, because it is analytic on any punctured open disc centred at 0. Now, earlier we saw that the entire function

$$g(z) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots$$

satisfies  $g(z) = (\sin z)/z$ , for  $z \neq 0$ . It follows that we can write

$$f(z) = \frac{g(z)}{z} \quad (z \in \mathbb{C} - \{0\}).$$

Since  $g(0) = 1$ , we can see that, informally speaking,  $f(z)$  behaves in the same way as  $1/z$  as  $z$  tends to 0. In particular,

$$f(z) \rightarrow \infty \text{ as } z \rightarrow 0.$$

Thus  $f$  does not have a removable singularity at 0; instead it is said to have a *simple pole* at 0.

### Definition

Let  $f$  be a function that has a singularity at the point  $\alpha$ .

Then  $f$  has a **simple pole** at  $\alpha$  if there is a function  $g$  that is analytic on an open disc  $\{z : |z - \alpha| < r\}$  such that  $g(\alpha) \neq 0$  and

$$f(z) = \frac{g(z)}{z - \alpha}, \quad \text{for } 0 < |z - \alpha| < r.$$

The expression  $f(z)$  behaves in the same way as  $g(\alpha)/(z - \alpha)$  as  $z$  tends to  $\alpha$ . In particular,

$$f(z) \rightarrow \infty \text{ as } z \rightarrow \alpha.$$

However, if we multiply  $f(z)$  by  $(z - \alpha)$  to obtain  $(z - \alpha)f(z) = g(z)$ , then in a sense we ‘cancel’ the simple pole at  $\alpha$ , and

$$\lim_{z \rightarrow \alpha} (z - \alpha)f(z) = \lim_{z \rightarrow \alpha} g(z) = g(\alpha).$$

These remarks are summarised in the observation below, which is a special case of Theorem 3.2, to come later.

### Observation

Let  $f$  be a function that has a simple pole at the point  $\alpha$ . Then

- (a)  $f(z) \rightarrow \infty$  as  $z \rightarrow \alpha$
- (b)  $\lim_{z \rightarrow \alpha} (z - \alpha)f(z)$  exists and is non-zero.

The next example returns to the function of Example 1.2.

### Example 1.3

Locate the simple poles of the function

$$f(z) = \frac{z + i}{z^2 + 1}.$$

**Solution**

We saw in Example 1.2 that  $f$  has singularities at  $-i$  and  $i$ , and  $-i$  is a removable singularity.

We also found that

$$f(z) = \frac{1}{z-i}, \quad \text{for } z \in \mathbb{C} - \{-i, i\}.$$

Hence

$$f(z) = \frac{g(z)}{z-i}, \quad \text{for } 0 < |z-i| < 2,$$

where  $g$  is the entire function  $g(z) = 1$ . Therefore  $f$  has a simple pole at  $i$ .

For the function  $f$  in Example 1.3, we have observed already (at the end of Example 1.2) that  $f(z) \rightarrow \infty$  as  $z \rightarrow i$ . In addition,

$$\lim_{z \rightarrow i} (z-i)f(z) = 1.$$

Therefore assertions (a) and (b) of the observation preceding Example 1.3 are both satisfied, as expected.

**Exercise 1.6**

Locate the simple poles of each of the following functions.

$$(a) \ f(z) = \frac{z-2}{z+1} \quad (b) \ f(z) = \frac{\cos z}{z} \quad (c) \ f(z) = \frac{z}{\sin z}$$

(Hint: For part (c), note that

$$f(z) = \frac{(-1)^k z}{\sin(z - k\pi)}, \quad \text{for } k \in \mathbb{Z},$$

since  $\sin(z - k\pi) = \sin z \cos k\pi - \cos z \sin k\pi = (-1)^k \sin z$ .)

Another way of describing a simple pole is to call it a *pole of order 1*. Let us now define poles in greater generality, motivating this generalisation by considering the function

$$f(z) = \frac{\sin z}{z^3} \quad (z \in \mathbb{C} - \{0\}),$$

which has a singularity at 0. We can write

$$f(z) = \frac{g(z)}{z^2} \quad (z \in \mathbb{C} - \{0\}),$$

where  $g$  is the entire function

$$g(z) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots \quad (z \in \mathbb{C}).$$

Then  $f(z) \rightarrow \infty$  as  $z \rightarrow 0$ , and, since  $g(0) = 1$ , we see that  $f(z)$  behaves in the same way as  $1/z^2$  as  $z$  tends to 0. In these circumstances,  $f$  is said to have a *pole of order 2* at 0.

The general definition of a pole follows.

### Definition

Let  $f$  be a function that has a singularity at the point  $\alpha$ .

Then  $f$  has a **pole of order  $k$**  at  $\alpha$  if there is a function  $g$  that is analytic on an open disc  $\{z : |z - \alpha| < r\}$  such that  $g(\alpha) \neq 0$  and

$$f(z) = \frac{g(z)}{(z - \alpha)^k}, \quad \text{for } 0 < |z - \alpha| < r.$$

Since

$$\lim_{z \rightarrow \alpha} (z - \alpha)^k f(z) = \lim_{z \rightarrow \alpha} g(z) = g(\alpha),$$

we can make the following observation about poles of order  $k$  (which is a special case of Theorem 3.2, to come later).

### Observation

Let  $f$  be a function that has a pole of order  $k$  at the point  $\alpha$ . Then

- (a)  $f(z) \rightarrow \infty$  as  $z \rightarrow \alpha$
- (b)  $\lim_{z \rightarrow \alpha} (z - \alpha)^k f(z)$  exists and is non-zero.

The next example is about classifying poles.

### Example 1.4

Locate the poles of the function

$$f(z) = \frac{z + i}{(z^2 + 1)^3},$$

and find their orders.

### Solution

The function  $f$  has singularities at  $-i$  and  $i$  (when  $z^2 + 1 = 0$ ), and it is analytic on  $\mathbb{C} - \{-i, i\}$ .

Observe that  $z^2 + 1 = (z + i)(z - i)$ , so we can write

$$f(z) = \frac{1}{(z + i)^2(z - i)^3}, \quad \text{for } z \in \mathbb{C} - \{-i, i\}.$$

First we classify the singularity at  $-i$ . Let

$$g(z) = \frac{1}{(z-i)^3} \quad (z \in \mathbb{C} - \{i\}).$$

Then  $g$  is analytic on the disc  $\{z : |z+i| < 2\}$  (with centre  $-i$  and radius 2) and

$$f(z) = \frac{g(z)}{(z+i)^2}, \quad \text{for } 0 < |z+i| < 2.$$

Since  $g(-i) \neq 0$ , we see that  $f$  has a pole of order 2 at  $-i$ .

Next we classify the singularity at  $i$ . This time, we define

$$g(z) = \frac{1}{(z+i)^2} \quad (z \in \mathbb{C} - \{-i\}).$$

This function  $g$  is analytic on the disc  $\{z : |z-i| < 2\}$  and

$$f(z) = \frac{g(z)}{(z-i)^3}, \quad \text{for } 0 < |z-i| < 2.$$

Since  $g(i) \neq 0$ , we see that  $f$  has a pole of order 3 at  $i$ .

### Exercise 1.7

Locate the poles of each of the following functions, and find their orders.

$$(a) \ f(z) = \frac{z+2}{z^4(z^2-4)^3} \quad (b) \ f(z) = \frac{z}{\sin^3 z}$$

We come now to our final type of singularity, which we explain by considering the function

$$f(z) = \sin(1/z) \quad (z \in \mathbb{C} - \{0\}).$$

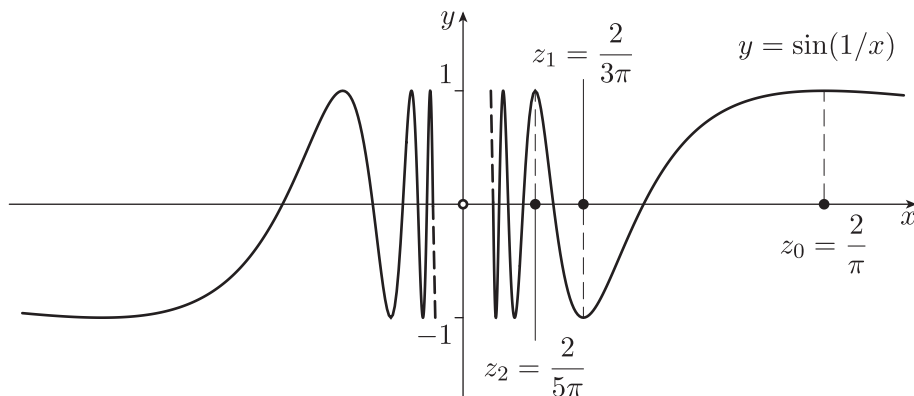
To explore the behaviour of this function near 0, recall once more that the Taylor series about 0 for  $\sin$  is

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots, \quad \text{for } z \in \mathbb{C}.$$

Substituting  $1/z$  for  $z$ , we obtain

$$f(z) = \frac{1}{z} - \frac{1}{3!z^3} + \frac{1}{5!z^5} - \cdots, \quad \text{for } z \in \mathbb{C} - \{0\}.$$

The form of this series suggests that  $f$  has neither a removable singularity nor a pole at 0. This suggestion is supported by the graph of  $y = \sin(1/x)$  for non-zero real values of  $x$ , shown in Figure 1.6.



**Figure 1.6** Graph of  $y = \sin(1/x)$

The graph oscillates infinitely often near the origin, which indicates that  $f(z)$  does not tend to a limit or to  $\infty$  as  $z$  tends to 0. In fact, we can prove this by looking at the sequence

$$z_n = \frac{2}{(2n+1)\pi}, \quad n = 0, 1, 2, \dots,$$

which tends to 0 as  $n$  tends to  $\infty$ . Observe that

$$f(z_n) = \sin(1/z_n) = \sin((n+1/2)\pi) = (-1)^n, \quad n = 0, 1, 2, \dots$$

Therefore the sequence  $(f(z_n))$  does not tend to a finite limit or to  $\infty$  as  $n$  tends to  $\infty$ , which implies that  $f$  has neither a removable singularity nor a pole at 0. In this case we say that  $f$  has an *essential singularity* at 0.

### Definition

Let  $f$  be a function that has a singularity at the point  $\alpha$ .

Then  $f$  has an **essential singularity** at  $\alpha$  if the singularity at  $\alpha$  is neither a removable singularity nor a pole.

We have seen that if  $f$  has a removable singularity at  $\alpha$ , then  $f(z)$  tends to a non-zero limit as  $z \rightarrow \alpha$ , and if  $f$  has a pole at  $\alpha$ , then  $f(z)$  tends to  $\infty$  as  $z \rightarrow \alpha$ . This gives us the following theorem for identifying essential singularities.

### Theorem 1.2

Let  $f$  be a function that has a singularity at the point  $\alpha$ . If  $f(z)$  does not tend to a finite limit or to  $\infty$  as  $z$  tends to  $\alpha$ , then  $f$  has an essential singularity at  $\alpha$ .

The next exercise gives you practice at recognising essential singularities.

**Exercise 1.8**

Locate the essential singularities (if any) of each of the following functions.

$$(a) f(z) = \frac{1}{(z-1)(z-3)^2} \quad (b) f(z) = e^{1/z}$$

$$(c) f(z) = \frac{z+i}{z(z^2+2iz-1)}$$

In Sections 2 and 3 we obtain alternative ways of classifying the three types of singularities.

**Further exercises****Exercise 1.9**

Locate the singularities of each of the following functions, and classify each singularity as a removable singularity, a pole (stating its order) or an essential singularity.

$$(a) f(z) = \frac{1}{z^5(z-i)^2} \quad (b) f(z) = \frac{z^2+1}{z(z-i)} \quad (c) f(z) = \frac{\sinh z}{z^4}$$

$$(d) f(z) = \sinh \frac{1}{z} \quad (e) f(z) = \frac{e^z - 1}{z} \quad (f) f(z) = e^{z-1}$$

$$(g) f(z) = \cot z \quad (h) f(z) = \frac{1}{e^z - 1}$$

(Hint: For part (d), use  $i \sinh z = \sin(iz)$  and the earlier discussion of  $f(z) = \sin(1/z)$ . For part (h), use  $e^z = e^{z-2k\pi i}$ , for  $k \in \mathbb{Z}$ .)

**2 Laurent's Theorem**

After working through this section, you should be able to:

- explain what is meant by the terms *extended power series*, *analytic part*, *singular part* and *Laurent series*
- understand the statement of Laurent's Theorem
- calculate the Laurent series of a given function that is analytic on an open annulus
- distinguish between the Laurent series about a removable singularity, a pole and an essential singularity.

## 2.1 Statement of Laurent's Theorem

Consider the functions

$$f_1(z) = \frac{\sin z}{z^3} \quad \text{and} \quad f_2(z) = \sin\left(\frac{1}{z}\right),$$

which you met in Subsection 1.3. You saw in that subsection that  $f_1$  and  $f_2$  can be represented by series involving *negative* powers of  $z$ , as follows:

$$f_1(z) = \frac{1}{z^2} - \frac{1}{3!} + \frac{z^2}{5!} - \frac{z^4}{7!} + \cdots, \quad \text{for } z \in \mathbb{C} - \{0\},$$

which involves only one negative power of  $z$  (namely  $z^{-2}$ ), and

$$f_2(z) = \frac{1}{z} - \frac{1}{3!z^3} + \frac{1}{5!z^5} - \cdots, \quad \text{for } z \in \mathbb{C} - \{0\},$$

which involves infinitely many negative powers ( $z^{-1}, z^{-3}, z^{-5}, \dots$ ).

Because of the presence of negative powers of  $z$ , these series are not ordinary power series; instead we refer to them as *extended power series*.

### Definitions

Let  $z \in \mathbb{C}$ . An expression of the form

$$\sum_{n=-\infty}^{\infty} a_n(z - \alpha)^n = \cdots + \frac{a_{-2}}{(z - \alpha)^2} + \frac{a_{-1}}{(z - \alpha)} + a_0 + a_1(z - \alpha) + \cdots,$$

where  $\alpha \in \mathbb{C}$  and  $a_n \in \mathbb{C}$ , for  $n \in \mathbb{Z}$ , is called an **extended power series about  $\alpha$** .

For a given  $z$ , the extended power series **converges** if the series

$$\sum_{n=0}^{\infty} a_n(z - \alpha)^n = a_0 + a_1(z - \alpha) + a_2(z - \alpha)^2 + \cdots$$

and

$$\sum_{n=1}^{\infty} a_{-n}(z - \alpha)^{-n} = \frac{a_{-1}}{(z - \alpha)} + \frac{a_{-2}}{(z - \alpha)^2} + \cdots$$

both converge.

These two series are called the **analytic part** and **singular part** of the extended power series, respectively.

At a point  $z$  for which the extended power series converges, we can form the **sum** of the extended power series at  $z$  by adding the sums of the analytic and singular parts:

$$\sum_{n=-\infty}^{\infty} a_n(z - \alpha)^n = \sum_{n=0}^{\infty} a_n(z - \alpha)^n + \sum_{n=1}^{\infty} a_{-n}(z - \alpha)^{-n}.$$

Returning to the examples from earlier in the subsection, we see that the

series arising from the function  $f_1(z) = (\sin z)/z^3$  has analytic part

$$-\frac{1}{3!} + \frac{z^2}{5!} - \frac{z^4}{7!} + \cdots$$

and singular part  $1/z^2$ . Also, the series arising from the function  $f_2(z) = \sin(1/z)$  has analytic part 0 and singular part

$$\frac{1}{z} - \frac{1}{3!z^3} + \frac{1}{5!z^5} - \cdots.$$

It is usual to write an extended power series that has *only* negative powers, like this one, in decreasing order of powers, as we have done here.

Notice that any ordinary power series is an extended power series with singular part 0.

We can define the *sum function* of an extended power series in the same way that we defined the sum function of a power series. That is, we let

$$A = \left\{ z : \sum_{n=-\infty}^{\infty} a_n(z - \alpha)^n \text{ converges} \right\},$$

and define the **sum function** of the extended power series to be the function

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z - \alpha)^n \quad (z \in A).$$

### Exercise 2.1

Express each of the following functions as an extended power series about 0, and identify the analytic and singular parts.

$$(a) \ f(z) = \frac{e^{2z}}{z^4} \quad (b) \ f(z) = e^z - e^{1/z} \quad (c) \ f(z) = \frac{\sin z}{z}$$

Next we investigate where extended power series converge. The key to this is to look separately at the analytic and singular parts. The analytic part converges (and defines an analytic function) on its disc of convergence

$$\{z : |z - \alpha| < s\}.$$

The singular part, which we assume here to be non-zero, can be thought of as an ordinary power series composed with  $(z - \alpha)^{-1}$ , so it converges when

$$|z - \alpha|^{-1} = |(z - \alpha)^{-1}| < s',$$

for some number  $s'$ . Let  $r = 1/s'$ . Then

$$|z - \alpha|^{-1} < s' \iff |z - \alpha| > r,$$

so the singular part converges (and defines an analytic function, by the Chain Rule) on

$$\{z : |z - \alpha| > r\}.$$

Thus both the analytic part and the singular part converge (and define analytic functions) on the intersection

$$\begin{aligned} A &= \{z : |z - \alpha| < s\} \cap \{z : |z - \alpha| > r\} \\ &= \{z : r < |z - \alpha| < s\}. \end{aligned}$$

This set may take any one of the following forms:

- (i) an open annulus ( $0 < r < s < \infty$ )
- (ii) a punctured open disc ( $r = 0, s < \infty$ )
- (iii) a punctured plane ( $r = 0, s = \infty$ )
- (iv) the outside of a closed disc ( $0 < r < s = \infty$ )
- (v) the empty set ( $r \geq s$ ).

Here we write  $r = 0$  when  $s' = \infty$ , and the statements  $s = \infty$  and  $s' = \infty$  are to be interpreted in the same way as the statement  $R = \infty$  that was used in Subsection 2.1 of Unit B3 when discussing the radius of convergence of a power series.

In each case the set  $A$  is an open set and is called the **annulus of convergence** of the extended power series. The series converges at all points inside  $A$ , and it diverges at all points that lie outside  $A$  and not on the boundary of  $A$ , because either the analytic part or the singular part of the series diverges at these points. The extended power series may or may not converge at points on the boundary of  $A$ .

Some of cases (i) to (v) are illustrated in Example 2.1 and Exercise 2.2.

### Example 2.1

Find the annulus of convergence of each of the following extended power series.

- (a)  $\cdots + \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + 1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \cdots$
- (b)  $\cdots + \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + 1 + z + z^2 + z^3 + \cdots$

### Solution

- (a) The analytic part is

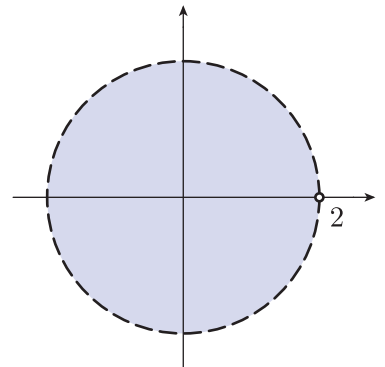
$$1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \cdots = 1 + \left(\frac{z}{2}\right) + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \cdots.$$

This series converges when  $|z/2| < 1$ , that is, for  $|z| < 2$  (see Figure 2.1).

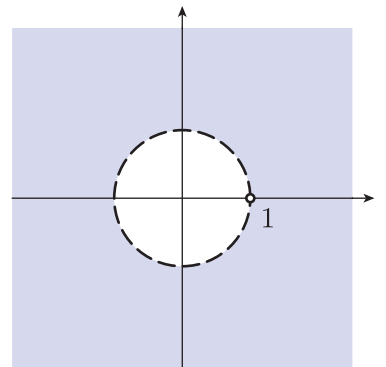
The singular part is

$$\frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots = \left(\frac{1}{z}\right) + \left(\frac{1}{z}\right)^2 + \left(\frac{1}{z}\right)^3 + \cdots.$$

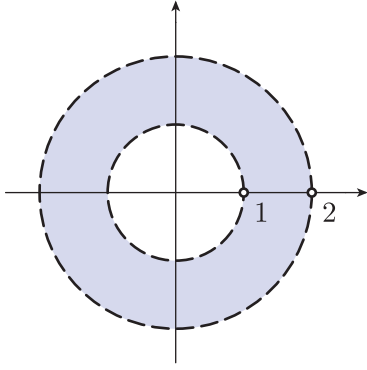
This series converges when  $0 < |1/z| < 1$ , that is, for  $|z| > 1$  (see Figure 2.2).



**Figure 2.1** The open disc  $\{z : |z| < 2\}$



**Figure 2.2** The open set  $\{z : |z| > 1\}$



**Figure 2.3** The open annulus  $\{z : 1 < |z| < 2\}$

Thus the annulus of convergence of

$$\cdots + \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + 1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \cdots$$

is  $\{z : 1 < |z| < 2\}$  (see Figure 2.3).

(b) The analytic part is

$$1 + z + z^2 + z^3 + \cdots.$$

This series converges for  $|z| < 1$ .

The singular part converges for  $|z| > 1$  (see part (a)).

Therefore the annulus of convergence of

$$\cdots + \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + 1 + z + z^2 + z^3 + \cdots$$

is  $\{z : |z| < 1\} \cap \{z : |z| > 1\} = \emptyset$ .

### Exercise 2.2

Find the annulus of convergence of each of the following extended power series.

(a)  $1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots$

(b)  $\frac{1}{z^2} + \frac{1}{z} + 1 + z + z^2 + z^3 + \cdots$

We stated above that the extended power series

$$\sum_{n=-\infty}^{\infty} a_n(z - \alpha)^n$$

defines an analytic function  $f$  on its annulus of convergence

$A = \{z : r < |z - \alpha| < s\}$ . The next theorem, which is the main result of this unit (and is proved in Subsection 2.3), is essentially a converse of this remark. Just as an analytic function can be represented by a Taylor series on any open disc in its domain, so a function that is analytic on an open annulus can be represented by an extended power series called a *Laurent series* (where Laurent is pronounced ‘luh-rawn’, or similar), which converges at all points in the annulus.

### Theorem 2.1 Laurent's Theorem

Let  $f$  be a function that is analytic on the open annulus

$$A = \{z : r < |z - \alpha| < s\}, \quad \text{where } 0 \leq r < s \leq \infty.$$

Then

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z - \alpha)^n, \quad \text{for } z \in A,$$

where

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - \alpha)^{n+1}} dz, \quad \text{for } n \in \mathbb{Z},$$

and  $C$  is any circle lying in  $A$  with centre  $\alpha$ .

Moreover, this representation of  $f$  on  $A$  as an extended power series about  $\alpha$  is unique.

Note the conventional use of ' $s \leq \infty$ ' in the equation for  $A$ , which allows  $A$  to be, for example, a punctured plane when  $r = 0$  and  $s = \infty$  (in which case, strictly speaking,  $A$  is not actually an annulus, because it is not a set bounded by two concentric circles).

It can also be shown, although we do not do so here, that under the hypotheses of Laurent's Theorem, the derivative  $f'$  is

$$f'(z) = \sum_{n=-\infty}^{\infty} n a_n (z - \alpha)^{n-1}, \quad \text{for } z \in A,$$

obtained by differentiating term by term, as you would expect.

### Definition

The representation

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z - \alpha)^n, \quad \text{for } z \in A,$$

determined by Laurent's Theorem is called the **Laurent series about  $\alpha$  for the function  $f$  on the annulus  $A$** .

Later in the module we will often use Laurent series in the case when  $A$  is a punctured open disc. We refer to such a series as the **Laurent series about  $\alpha$  for the function  $f$**  (without explicit mention of the set  $A$ ).

## Remarks

1. Notice that the formula

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - \alpha)^{n+1}} dz, \quad \text{for } n \in \mathbb{Z}, \quad (2.1)$$

for the coefficients of the Laurent series about  $\alpha$  is the same as that for the coefficients of the Taylor series about  $\alpha$ , when  $n$  is non-negative (see the remark after the definition of Taylor series in Subsection 3.1 of Unit B3). Here, the formula is equally valid for *negative* values of  $n$ .

For Taylor series, we also have

$$a_n = f^{(n)}(\alpha)/n!, \quad \text{for } n = 0, 1, 2, \dots,$$

but for Laurent series, we cannot always write  $a_n$  in this way since  $f$  may not be differentiable at  $\alpha$ .

2. The uniqueness statement in Laurent's Theorem is important. It implies that *any* given representation of an analytic function  $f$  on an annulus  $A$  by an extended power series must be the Laurent series for  $f$  on  $A$ , with coefficients given by equation (2.1). For example, if  $f(z) = (\sin z)/z^3$ , then

$$f(z) = \frac{1}{z^2} - \frac{1}{3!} + \frac{z^2}{5!} - \frac{z^4}{7!} + \dots, \quad \text{for } z \in \mathbb{C} - \{0\},$$

as we saw at the start of this subsection. This is an extended power series about 0 that represents  $f$ , so it must be the Laurent series about 0 for  $f$ .

3. If  $f$  is analytic on the disc  $\{z : |z - \alpha| < r\}$ , then the Laurent series about  $\alpha$  for  $f$  on the punctured disc  $\{z : 0 < |z - \alpha| < r\}$  has the same terms as the Taylor series about  $\alpha$  for  $f$ .
4. It is important to note that a Laurent series depends on the annulus  $A$ . For example, the function

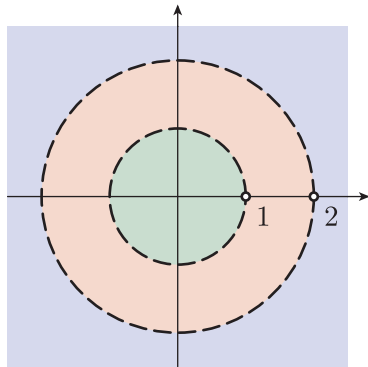
$$f(z) = \frac{1}{(z - 1)(z - 2)}$$

is analytic on  $\mathbb{C} - \{1, 2\}$ , so it is analytic on each of the three sets

$$\{z : |z| < 1\}, \quad \{z : 1 < |z| < 2\}, \quad \{z : |z| > 2\},$$

illustrated in Figure 2.4. The function  $f$  has *three different* Laurent series about 0, one on each of these sets. We return to this example in Subsection 2.2.

5. Laurent series may converge at points of the boundary of the annulus of convergence or they may not.



**Figure 2.4** Three sets determined by  $|z| < 1$ ,  $1 < |z| < 2$  and  $|z| > 2$

## Exercise 2.3

By using the Taylor series about 0 for the functions  $\exp$  and  $\sinh$ , write down the Laurent series about 0 for each of the following functions.

(a)  $f(z) = e^{1/z}$       (b)  $f(z) = \frac{\sinh 2z}{z^2}$

**Remark**

When asked for a Laurent series of a function, you must choose which terms of the series to write down in order to clearly represent the series. Here we follow the same sorts of conventions that we used for Taylor series in the preceding unit: where possible, you should include enough terms to make it clear how the sequence of coefficients of the Laurent series continues. Unlike Taylor series, Laurent series can have positive and negative powers of  $z$ , so you may need to judge how many terms to include in the analytic and singular parts of the Laurent series, separately. Usually it should be clear how to make a sensible choice of which terms to include, and the examples in the unit will guide you.

We conclude this subsection by stating a result that relates the type of singularity at  $\alpha$  of a function  $f$  to the Laurent series about  $\alpha$  for  $f$ . In order to introduce it, we recall three examples from Subsection 1.3.

The function  $f(z) = (\sin z)/z$  has a removable singularity at 0; its Laurent series about 0 is

$$\frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots, \quad \text{for } z \in \mathbb{C} - \{0\}.$$

The function  $f(z) = (\sin z)/z^3$  has a pole of order 2 at 0; its Laurent series about 0 is

$$\frac{\sin z}{z^3} = \frac{1}{z^2} - \frac{1}{3!} + \frac{z^2}{5!} - \cdots, \quad \text{for } z \in \mathbb{C} - \{0\}.$$

The function  $f(z) = \sin(1/z)$  has an essential singularity at 0; its Laurent series about 0 is

$$\sin\left(\frac{1}{z}\right) = \frac{1}{z} - \frac{1}{3!z^3} + \frac{1}{5!z^5} - \cdots, \quad \text{for } z \in \mathbb{C} - \{0\}.$$

These three functions illustrate the three parts of the following theorem, which is proved in Subsection 2.3.

**Theorem 2.2**

Let  $f$  be a function that has a singularity at the point  $\alpha$ , and suppose that the Laurent series about  $\alpha$  for  $f$  is

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z - \alpha)^n.$$

Then

- (a)  $f$  has a removable singularity at  $\alpha$  if and only if  
 $a_n = 0$  for all  $n < 0$
- (b)  $f$  has a pole of order  $k \in \mathbb{N}$  at  $\alpha$  if and only if  
 $a_{-k} \neq 0$  and  $a_n = 0$  for all  $n < -k$
- (c)  $f$  has an essential singularity at  $\alpha$  if and only if  
 $a_n \neq 0$  for infinitely many  $n < 0$ .

Some texts use the classification (a)–(c) of singularities to *define* the three types of singularities.

Proving that a function  $f$  has an essential singularity at a point by using Theorem 2.2(c) involves less work than by using Theorem 1.2. For example, consider the function  $f(z) = e^{1/z}$ . The Laurent series about 0 for  $f$  is

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots, \quad \text{for } |z| > 0,$$

which has infinitely many non-zero coefficients in its singular part. Hence, by Theorem 2.2(c),  $f$  has an essential singularity at 0. This is much shorter than the solution to Exercise 1.8(b), in which we used Theorem 1.2.

### Exercise 2.4

Show that the function  $f(z) = z \sin(1/z^2)$  has an essential singularity at 0.



Karl Theodor Wilhelm Weierstrass

### Weierstrass's legacy

The earliest discoveries in complex analysis were made by Cauchy and his contemporaries, but it was the work of the German mathematician Karl Theodor Wilhelm Weierstrass (1815–1897) and his followers that placed the subject on firm foundations.

Weierstrass (pronounced ‘vye-uh-strass’) was an unusual mathematician in that he published little, choosing to deliver his findings in university lecture courses. He rejected many of Cauchy’s methods, and criticised Riemann’s geometric approach. Instead he championed the use of power series as the objects on which to build complex analysis. In a talk presented in 1884, years after many of his discoveries, he gave the following explanation for his methods.

However great the importance of the notion of the integral for all of analysis, I nevertheless wish to found the theory of functions solely with the help of the elementary theorems about the basic operations. I do not say that one should give, or can give such a direct proof in each case; I leave this question undecided. But I try to give direct proofs as far as possible, and I want to use this method particularly with the foundation of the theory of functions.

(Bottazzini and Gray, 2013, p. 352)

Weierstrass was the first to appreciate the distinction between poles and essential singularities, and he was probably also the first to prove Laurent’s Theorem. This theorem was stated in a letter submitted in 1843 to the Académie des Sciences of France by Pierre Alphonse Laurent (1813–1854), a French mathematician and military officer.

The letter was a fragment of a longer paper, which the Académie refused to publish, and within a few years Laurent's interests shifted towards applied mathematics. Nonetheless, the significance of the theorem was recognised by Cauchy, who set about explaining how it could be obtained from his own earlier work. However, it seems that Weierstrass knew of the theorem already, because it appears with a proof in one of his papers from 1841, which he did not publish at the time.

Despite publishing little, Weierstrass's influence was substantial through the work of his many students. Today he is rightly regarded alongside Cauchy and Riemann as one of the founders of complex analysis.

## 2.2 Calculating Laurent series

In this subsection we calculate the Laurent series for various functions that are analytic on an annulus. In the previous subsection several Laurent series were obtained by modifying appropriate Taylor series (see Exercise 2.3, for example), and this is often the easiest method for determining a Laurent series. It is rarely a good idea to obtain a Laurent series by calculating the coefficients  $a_n$  using the formula

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - \alpha)^{n+1}} dz.$$

Indeed, later we evaluate integrals of this type by using the associated Laurent series.

As we mentioned earlier, the rational function

$$f(z) = \frac{1}{(z-1)(z-2)}$$

is analytic on each of the three annuli

$$\{z : |z| < 1\}, \quad \{z : 1 < |z| < 2\}, \quad \{z : |z| > 2\}.$$

In order to find the Laurent series about 0 for  $f$  on each of these annuli, we start by expressing  $f$  in partial fractions as

$$f(z) = \frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1},$$

and then work with each partial fraction independently.

Each of these partial fractions involves a function of the form

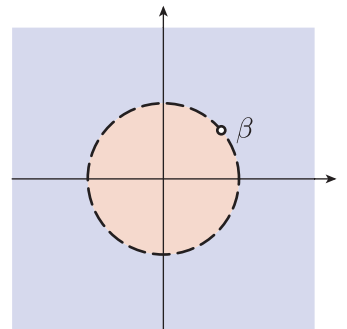
$$g(z) = \frac{1}{z - \beta},$$

where  $\beta \neq 0$ , so let us find the Laurent series about 0 for  $g$ .

The function  $g$  is analytic on  $\mathbb{C} - \{\beta\}$ , so it is analytic on each of the sets

$$\{z : |z| < |\beta|\} \quad \text{and} \quad \{z : |z| > |\beta|\}$$

shown in Figure 2.5.



**Figure 2.5** The sets  $\{z : |z| < |\beta|\}$  and  $\{z : |z| > |\beta|\}$

The Laurent series about 0 for  $g$  on each of these sets is found by rearranging  $1/(z - \beta)$  and using the formula for geometric series, as follows. First suppose that  $|z| < |\beta|$ , so  $|z/\beta| < 1$ . Then

$$\begin{aligned} g(z) &= \frac{1}{z - \beta} \\ &= -\frac{1}{\beta} \times \frac{1}{1 - z/\beta} \\ &= -\frac{1}{\beta} \left( 1 + \frac{z}{\beta} + \left( \frac{z}{\beta} \right)^2 + \cdots \right) \\ &= -\frac{1}{\beta} - \frac{z}{\beta^2} - \frac{z^2}{\beta^3} - \cdots, \quad \text{for } |z| < |\beta|. \end{aligned} \quad (2.2)$$

Next suppose that  $|z| > |\beta|$ , so  $|\beta/z| < 1$ . Then

$$\begin{aligned} g(z) &= \frac{1}{z - \beta} \\ &= \frac{1}{z} \times \frac{1}{1 - \beta/z} \\ &= \frac{1}{z} \left( 1 + \frac{\beta}{z} + \left( \frac{\beta}{z} \right)^2 + \cdots \right) \\ &= \frac{1}{z} + \frac{\beta}{z^2} + \frac{\beta^2}{z^3} + \cdots, \quad \text{for } |z| > |\beta|. \end{aligned} \quad (2.3)$$

There is no need to remember equations (2.2) and (2.3), because you can calculate them afresh for any particular function  $g$  that you are working with. In fact, we will do so in the following example, where we complete our discussion of Laurent series about 0 for the function

$$f(z) = \frac{1}{(z - 1)(z - 2)}$$

considered earlier in this subsection.

### Example 2.2

Find the Laurent series about 0 for the function

$$f(z) = \frac{1}{(z - 1)(z - 2)}$$

on each of the following regions.

- (a)  $A = \{z : |z| < 1\}$       (b)  $B = \{z : 1 < |z| < 2\}$   
 (c)  $C = \{z : |z| > 2\}$

### Solution

Recall that

$$f(z) = \frac{1}{(z - 1)(z - 2)} = \frac{1}{z - 2} - \frac{1}{z - 1}.$$

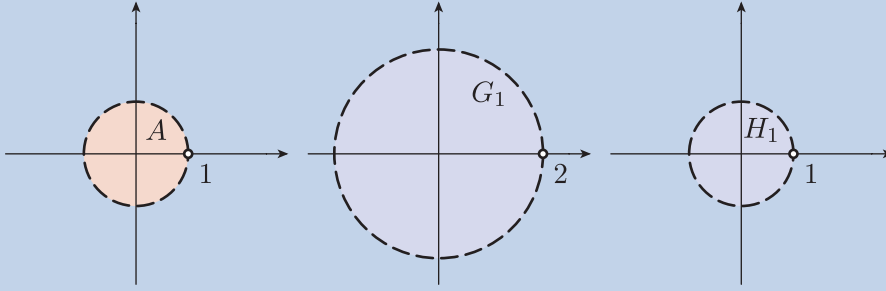
- (a) The function  $g(z) = 1/(z - 2)$  is analytic on

$$G_1 = \{z : |z| < 2\} \quad \text{and} \quad G_2 = \{z : |z| > 2\},$$

and the function  $h(z) = 1/(z - 1)$  is analytic on

$$H_1 = \{z : |z| < 1\} \quad \text{and} \quad H_2 = \{z : |z| > 1\}.$$

Thus, since  $A \subseteq G_1$  and  $A \subseteq H_1$  (in fact,  $A = H_1$ ), as illustrated in Figure 2.6, we can find the Laurent series for  $f$  on  $A$  by finding the Laurent series for  $g$  on  $G_1$  and for  $h$  on  $H_1$ , and subtracting them.



**Figure 2.6** The open discs  $A$ ,  $G_1$  and  $H_1$ , where  $A \subseteq G_1$  and  $A = H_1$

If  $z \in G_1$ , then  $|z| < 2$ , so  $|z/2| < 1$ . Hence

$$\begin{aligned} g(z) &= \frac{1}{z-2} \\ &= -\frac{1}{2} \times \frac{1}{1-z/2} \\ &= -\frac{1}{2} \left( 1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \cdots \right) \\ &= -\frac{1}{2} - \frac{z}{4} - \frac{z^2}{8} - \frac{z^3}{16} - \cdots, \quad \text{for } |z| < 2. \end{aligned}$$

If  $z \in H_1$ , then  $|z| < 1$ . Hence

$$\begin{aligned} h(z) &= \frac{1}{z-1} \\ &= -\frac{1}{1-z} \\ &= -(1 + z + z^2 + z^3 + \cdots) \\ &= -1 - z - z^2 - z^3 - \cdots, \quad \text{for } |z| < 1. \end{aligned}$$

As expected, the Laurent series for both  $g$  and  $h$  reduce to Taylor series about 0, because  $g$  is analytic on  $G_1$  and  $h$  is analytic on  $H_1$ .

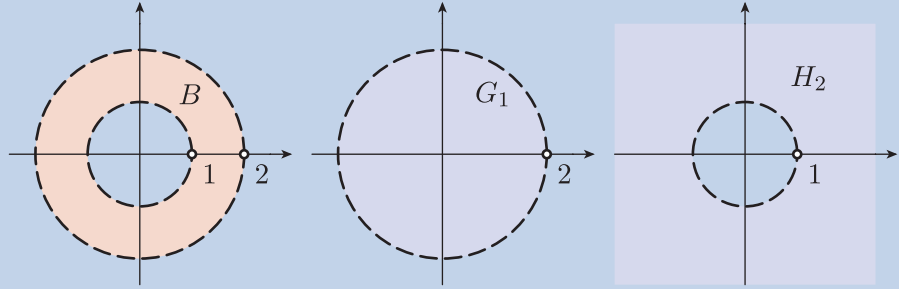
These power series are both valid for  $|z| < 1$ , so

$$\begin{aligned} f(z) &= g(z) - h(z) \\ &= \left( -\frac{1}{2} - \frac{z}{4} - \frac{z^2}{8} - \frac{z^3}{16} - \cdots \right) - (-1 - z - z^2 - z^3 - \cdots) \\ &= \frac{1}{2} + \frac{3}{4}z + \frac{7}{8}z^2 + \frac{15}{16}z^3 + \cdots, \quad \text{for } |z| < 1. \end{aligned}$$

(b) Observe that  $B = \{z : 1 < |z| < 2\}$  is such that

$$B \subseteq G_1 \quad \text{and} \quad B \subseteq H_2,$$

as shown in Figure 2.7.



**Figure 2.7** The open sets  $B$ ,  $G_1$  and  $H_2$ , where  $B \subseteq G_1$  and  $B \subseteq H_2$

The Laurent series for  $g$  on  $G_1$  was found in part (a) to be

$$g(z) = -\frac{1}{2} - \frac{z}{4} - \frac{z^2}{8} - \frac{z^3}{16} - \cdots, \quad \text{for } |z| < 2.$$

For the Laurent series for  $h$  on  $H_2$ , observe that if  $z \in H_2$ , then  $|z| > 1$ , so  $|1/z| < 1$ . Hence

$$\begin{aligned} h(z) &= \frac{1}{z-1} \\ &= \frac{1}{z} \times \frac{1}{1-1/z} \\ &= \frac{1}{z} \left( 1 + \frac{1}{z} + \left(\frac{1}{z}\right)^2 + \cdots \right) \\ &= \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots, \quad \text{for } |z| > 1. \end{aligned}$$

Since  $f(z) = g(z) - h(z)$ , we obtain

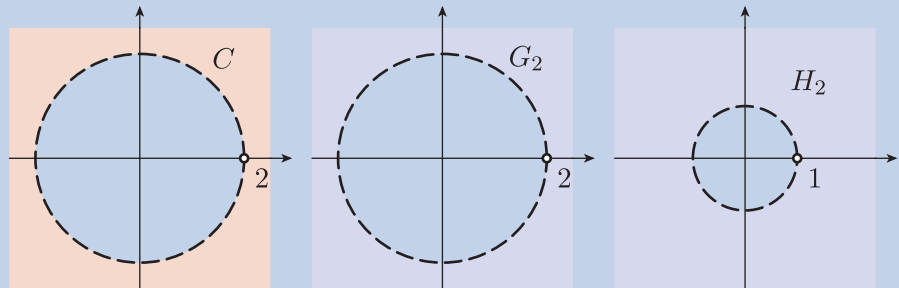
$$f(z) = \cdots - \frac{1}{z^3} - \frac{1}{z^2} - \frac{1}{z} - \frac{1}{2} - \frac{z}{4} - \frac{z^2}{8} - \frac{z^3}{16} - \cdots,$$

for  $1 < |z| < 2$ .

(c) Next,  $C = \{z : |z| > 2\}$  is such that

$$C = G_2 \quad \text{and} \quad C \subseteq H_2,$$

as shown in Figure 2.8.



**Figure 2.8** The open sets  $C$ ,  $G_2$  and  $H_2$ , where  $C = G_2$  and  $C \subseteq H_2$

For the Laurent series for  $g$  on  $G_2$ , observe that if  $z \in G_2$ , then  $|z| > 2$ , so  $|2/z| < 1$ . Hence

$$\begin{aligned} g(z) &= \frac{1}{z-2} \\ &= \frac{1}{z} \times \frac{1}{1-2/z} \\ &= \frac{1}{z} \left( 1 + \frac{2}{z} + \left(\frac{2}{z}\right)^2 + \left(\frac{2}{z}\right)^3 + \cdots \right) \\ &= \frac{1}{z} + \frac{2}{z^2} + \frac{4}{z^3} + \frac{8}{z^4} + \cdots, \quad \text{for } |z| > 2. \end{aligned}$$

The Laurent series for  $h$  on  $H_2$  was found in part (b) to be

$$h(z) = \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \cdots, \quad \text{for } |z| > 1.$$

Since  $f(z) = g(z) - h(z)$ , we obtain

$$f(z) = \frac{1}{z^2} + \frac{3}{z^3} + \frac{7}{z^4} + \cdots, \quad \text{for } |z| > 2.$$

### Remarks

1. Since  $f(z) = h(z)g(z)$ , we *could* have found each of the Laurent series for  $f$  by multiplying the appropriate series for  $h$  and  $g$ . In general, this procedure is not followed since it is more tedious to multiply Laurent series than it is to add or subtract them.
2. We have combined the Laurent series for  $g$  and  $h$  term by term, just as we would do for Taylor series. That we can add Laurent series in this way can be justified by adding the analytic parts and singular parts separately, using the Sum Rule for power series (Theorem 4.1(a) of Unit B3).
3. In Example 2.2(c) we obtained the Laurent series

$$f(z) = \frac{1}{z^2} + \frac{3}{z^3} + \frac{7}{z^4} + \cdots, \quad \text{for } |z| > 2.$$

We could have expressed this series in the form

$$f(z) = \frac{2^1 - 1}{z^2} + \frac{2^2 - 1}{z^3} + \frac{2^3 - 1}{z^4} + \cdots,$$

from which we can immediately see how the series continues, and we can find the general term. However, for simplicity, we do not usually present Laurent series in this way (we usually ‘simplify’ the coefficients), and we specify the general term only if we have a use for it.

4. As you become skilled at using the partial fractions approach, you may find that you can dispense with the explicit argument relating the annulus of convergence of  $f$  to the annuli associated with the functions appearing in the partial fractions.

**Exercise 2.5**

- (a) Determine how many different Laurent series the function

$$f(z) = \frac{1}{z(z-1)}$$

has about 0, and find their annuli of convergence.

- (b) Find the Laurent series about 0 for
- $f$
- on each of the annuli of convergence found in part (a).

(There is no need to use partial fractions here.)

**Exercise 2.6**

Find the Laurent series about 0 for the function

$$f(z) = \frac{4}{(z-1)(z+3)}$$

on each of the following regions.

- (a)
- $\{z : |z| < 1\}$
- (b)
- $\{z : 1 < |z| < 3\}$
- (c)
- $\{z : |z| > 3\}$

**Exercise 2.7**

Determine how many different Laurent series the function

$$f(z) = \frac{z-3i}{z(z-i)(z+3)(z-5)}$$

has about 0, and find their annuli of convergence.

(You are *not* required to find these Laurent series.)

Up to now we have concentrated exclusively on Laurent series about 0. To find a Laurent series about a point other than 0, the simplest approach is to make a substitution that enables us to obtain the required Laurent series from a Laurent series about 0 that we know how to calculate. This procedure is similar to the Substitution Rule for power series, discussed in Subsection 4.1 of Unit B3.

The following example should make the method clear.

**Example 2.3**

Find the Laurent series about 1 for each of the following functions.

$$(a) f(z) = \frac{e^z}{(z-1)^2} \quad (b) f(z) = \frac{1}{z(z-1)}$$

**Solution**

The question does not mention an annulus of convergence, which implies that, in each case, we must find the Laurent series on a punctured open disc with centre 1.

To do this, we define  $w = z - 1$  and express  $f(z)$  in terms of  $w$ . We then find the Laurent series about 0 for this expression in terms of  $w$ , and substitute  $w = z - 1$  to give the Laurent series about 1 for  $f$ .

- (a) We let  $w = z - 1$ , so  $z = 1 + w$ . Then, for  $z \neq 1$  (equivalently,  $w \neq 0$ ), we have

$$\begin{aligned} f(z) &= \frac{e^{1+w}}{w^2} \\ &= \frac{e}{w^2} \times e^w \\ &= \frac{e}{w^2} \left( 1 + w + \frac{w^2}{2!} + \frac{w^3}{3!} + \cdots \right) \\ &= \frac{e}{w^2} + \frac{e}{w} + \frac{e}{2} + \frac{e}{6}w + \cdots \\ &= \frac{e}{(z-1)^2} + \frac{e}{z-1} + \frac{e}{2} + \frac{e}{6}(z-1) + \cdots, \end{aligned}$$

for  $z \in \mathbb{C} - \{1\}$ .

In this case, the series converges on the punctured plane  $\mathbb{C} - \{1\}$ .

- (b) Again, we let  $w = z - 1$ , so  $z = 1 + w$ . Then, for  $z \neq 0, 1$  (equivalently,  $w \neq -1, 0$ ), we have

$$\begin{aligned} f(z) &= \frac{1}{(1+w)w} \\ &= \frac{1}{w} \times \frac{1}{1+w} \\ &= \frac{1}{w} (1 - w + w^2 - w^3 + \cdots), \quad \text{for } 0 < |w| < 1, \\ &= \frac{1}{w} - 1 + w - w^2 + \cdots \\ &= \frac{1}{z-1} - 1 + (z-1) - (z-1)^2 + \cdots, \end{aligned}$$

for  $0 < |z-1| < 1$ .

This series converges on the punctured disc  $\{z : 0 < |z-1| < 1\}$ .

**Exercise 2.8**

Find the Laurent series about 2 for each of the following functions.

(a)  $f(z) = \frac{\cos(z-2)}{(z-2)^2}$       (b)  $f(z) = z \cos\left(\frac{1}{z-2}\right)$       (c)  $f(z) = \frac{1}{z^2 - 4}$

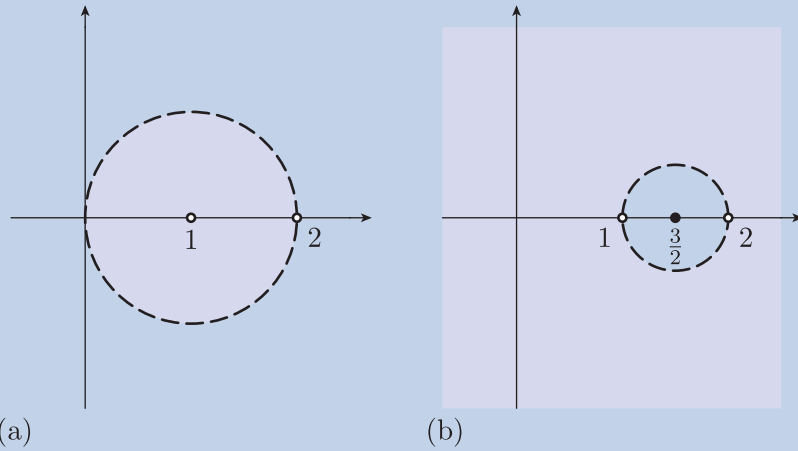
Finally, we return to the function in Example 2.2 and represent it as a Laurent series about two other points.

**Example 2.4**

Consider the function

$$f(z) = \frac{1}{(z-1)(z-2)}.$$

- (a) Find the Laurent series about the point 1 on the punctured disc  $\{z : 0 < |z-1| < 1\}$  illustrated in Figure 2.9(a).  
 (b) Find the Laurent series about the point  $\frac{3}{2}$  on the region  $\{z : |z - \frac{3}{2}| > \frac{1}{2}\}$  illustrated in Figure 2.9(b).



**Figure 2.9** (a) The punctured disc  $\{z : 0 < |z-1| < 1\}$  (b) The region  $\{z : |z - \frac{3}{2}| > \frac{1}{2}\}$

**Solution**

- (a) To find the Laurent series about 1, we define  $w = z - 1$  and express  $f(z)$  in terms of  $w$ . We then find the Laurent series about 0 for this expression in terms of  $w$  on the punctured disc  $\{w : 0 < |w| < 1\}$ . Substituting  $w = z - 1$  gives the required Laurent series about 1.

Let  $w = z - 1$ , so  $z = 1 + w$ . Then, for  $z \neq 1, 2$ ,

$$\begin{aligned} f(z) &= \frac{1}{w(w-1)} \\ &= -\frac{1}{w} \times \frac{1}{1-w} \\ &= -\frac{1}{w}(1 + w + w^2 + w^3 + \dots), \quad \text{for } 0 < |w| < 1, \\ &= -\frac{1}{w} - 1 - w - w^2 - \dots \\ &= -\frac{1}{z-1} - 1 - (z-1) - (z-1)^2 - \dots, \end{aligned}$$

for  $0 < |z-1| < 1$ .

- (b) We follow a similar strategy to part (a), but this time we substitute  $w = z - \frac{3}{2}$ , so  $z = \frac{3}{2} + w$ . Then  $z - 1 = w + \frac{1}{2}$  and  $z - 2 = w - \frac{1}{2}$ . Thus, for  $z \neq 1, 2$ ,

$$\begin{aligned} f(z) &= \frac{1}{(w + \frac{1}{2})(w - \frac{1}{2})} \\ &= \frac{1}{w^2 - \frac{1}{4}} \\ &= \frac{1}{w^2} \times \frac{1}{1 - 1/(4w^2)} \\ &= \frac{1}{w^2} \left( 1 + \left( \frac{1}{4w^2} \right) + \left( \frac{1}{4w^2} \right)^2 + \left( \frac{1}{4w^2} \right)^3 + \cdots \right), \end{aligned}$$

for  $|1/(4w^2)| < 1$ , or, equivalently,  $|w| > \frac{1}{2}$ . Hence

$$\begin{aligned} f(z) &= \frac{1}{w^2} + \frac{1}{4w^4} + \frac{1}{16w^6} + \cdots \\ &= \frac{1}{(z - \frac{3}{2})^2} + \frac{1}{4(z - \frac{3}{2})^4} + \frac{1}{16(z - \frac{3}{2})^6} + \cdots, \end{aligned}$$

for  $|z - \frac{3}{2}| > \frac{1}{2}$ .

For part (b) of Example 2.4 we could have split the expression  $1/((w + \frac{1}{2})(w - \frac{1}{2}))$  into partial fractions, and then expanded each resulting term separately. However, that method involves more algebraic manipulations.

### Exercise 2.9

Consider the function

$$f(z) = \frac{1}{z(z+3)(z+6)}.$$

Determine how many different Laurent series the function has about the following points, and find their annuli of convergence.

- (a)  $-3$       (b)  $-6$

### Exercise 2.10

Consider the function

$$f(z) = \frac{4}{(z-1)(z+3)}.$$

- (a) Find the Laurent series about the point  $-3$  on the punctured disc  $\{z : 0 < |z+3| < 4\}$ .  
 (b) Find the Laurent series about the point  $-1$  on the region  $\{z : |z+1| > 2\}$ .

## 2.3 Proofs of theorems on Laurent series

In this subsection we prove Theorem 2.1 (Laurent's Theorem) and Theorem 2.2. Some aspects of the proofs are challenging, so you may choose to skim through the details on a first reading.

### Theorem 2.1 Laurent's Theorem

Let  $f$  be a function that is analytic on the open annulus

$$A = \{z : r < |z - \alpha| < s\}, \quad \text{where } 0 \leq r < s \leq \infty.$$

Then

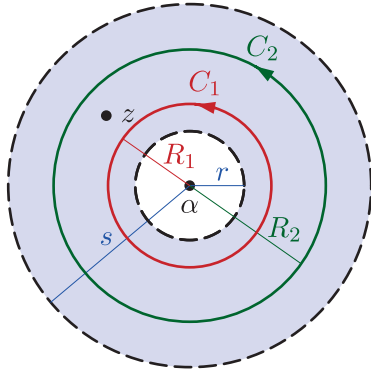
$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z - \alpha)^n, \quad \text{for } z \in A,$$

where

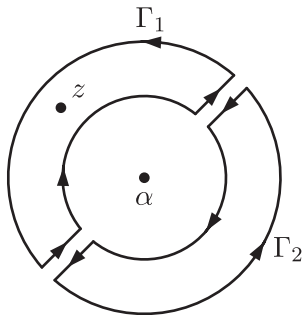
$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - \alpha)^{n+1}} dz, \quad \text{for } n \in \mathbb{Z},$$

and  $C$  is any circle lying in  $A$  with centre  $\alpha$ .

Moreover, this representation of  $f$  on  $A$  as an extended power series about  $\alpha$  is unique.



**Figure 2.10** The circles  $C_1$  and  $C_2$  inside  $A$



**Figure 2.11** The contours  $\Gamma_1$  and  $\Gamma_2$ , pulled apart slightly

**Proof** The proof is in five steps.

1. Let  $z$  be any point in  $A$ , let  $R_1$  and  $R_2$  be real numbers satisfying

$$r < R_1 < |z - \alpha| < R_2 < s,$$

and let  $C_1$  and  $C_2$  be the circles with centre  $\alpha$  and radii  $R_1$  and  $R_2$ , respectively (see Figure 2.10).

We first prove that

$$f(z) = \frac{1}{2\pi i} \int_{C_2} \frac{f(w)}{w - z} dw - \frac{1}{2\pi i} \int_{C_1} \frac{f(w)}{w - z} dw. \quad (2.4)$$

To obtain this formula we split the region between  $C_1$  and  $C_2$  into two parts by means of two line segments, and let  $\Gamma_1$  and  $\Gamma_2$  be the contours shown in Figure 2.11 (in which the contours have been 'pulled apart' slightly for clarity; really the two contours coincide along the line segments). The line segments are chosen so that  $\Gamma_1$  encloses  $z$ . Then, by Cauchy's Integral Formula (Theorem 2.1 of Unit B2),

$$\frac{1}{2\pi i} \int_{\Gamma_1} \frac{f(w)}{w - z} dw = f(z),$$

and, by Cauchy's Theorem (Theorem 1.2 of Unit B2),

$$\frac{1}{2\pi i} \int_{\Gamma_2} \frac{f(w)}{w - z} dw = 0.$$

If we add these results, then the integrals along the line segments cancel, and the semicircular parts of  $\Gamma_1$  and  $\Gamma_2$  combine to give  $C_1$  and  $C_2$ . This proves equation (2.4).

2. We now look separately at the two integrals on the right-hand side of equation (2.4). We will see that they give us, respectively, the analytic part and the singular part of the Laurent series.

We first note that

$$\frac{1}{2\pi i} \int_{C_2} \frac{f(w)}{w-z} dw = \sum_{n=0}^{\infty} a_n (z-\alpha)^n,$$

where

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-\alpha)^{n+1}} dz, \quad \text{for } n \in \mathbb{Z}. \quad (2.5)$$

To prove this we simply repeat steps 2–4 of the proof of Taylor's Theorem (Theorem 3.1 of Unit B3); the method is identical, although we cannot deduce here that  $a_n = f^{(n)}(\alpha)/n!$ , as  $f$  may not be analytic at all points inside  $C_2$ .

This gives the analytic part of the required series.

3. To obtain the singular part, we consider the other integral in equation (2.4), namely

$$-\frac{1}{2\pi i} \int_{C_1} \frac{f(w)}{w-z} dw.$$

First note that

$$-\frac{1}{w-z} = \frac{1}{(z-\alpha) - (w-\alpha)} = \frac{1}{z-\alpha} \left(1 - \frac{w-\alpha}{z-\alpha}\right)^{-1}.$$

But, for any  $\lambda \in \mathbb{C} - \{1\}$ , we have

$$(1-\lambda)^{-1} = 1 + \lambda + \lambda^2 + \cdots + \lambda^{n-1} + \frac{\lambda^n}{1-\lambda},$$

as you can check by multiplying both sides of the equation by  $1-\lambda$ .

Replacing  $\lambda$  by  $(w-\alpha)/(z-\alpha)$ , we obtain

$$\begin{aligned} -\frac{1}{w-z} &= \frac{1}{z-\alpha} \left( 1 + \frac{w-\alpha}{z-\alpha} + \cdots + \left( \frac{w-\alpha}{z-\alpha} \right)^{n-1} \right. \\ &\quad \left. + \left( \frac{w-\alpha}{z-\alpha} \right)^n \times \frac{1}{1 - (w-\alpha)/(z-\alpha)} \right). \end{aligned}$$

Hence

$$-\frac{1}{w-z} = \frac{1}{z-\alpha} + \frac{w-\alpha}{(z-\alpha)^2} + \cdots + \frac{(w-\alpha)^{n-1}}{(z-\alpha)^n} + \frac{(w-\alpha)^n}{(z-\alpha)^n} \frac{1}{(z-w)}.$$

Thus

$$\begin{aligned} &-\frac{1}{2\pi i} \int_{C_1} \frac{f(w)}{w-z} dw \\ &= \frac{1}{2\pi i} \int_{C_1} \left( \frac{1}{z-\alpha} + \frac{w-\alpha}{(z-\alpha)^2} + \cdots + \frac{(w-\alpha)^{n-1}}{(z-\alpha)^n} \right) f(w) dw \\ &\quad + \frac{1}{2\pi i} \int_{C_1} \frac{(w-\alpha)^n}{(z-\alpha)^n} \frac{f(w)}{z-w} dw. \end{aligned} \quad (2.6)$$

4. Let  $I_n$  be the last integral in equation (2.6). We now use the Estimation Theorem (Theorem 4.1 of Unit B1) to show that  $I_n \rightarrow 0$  as  $n \rightarrow \infty$ . Since  $f$  is analytic on  $A$ ,  $f$  is continuous on  $C_1$ , and thus  $f$  is bounded on  $C_1$ ; that is, there is a number  $K$  such that  $|f(w)| \leq K$ , for  $w \in C_1$ . Also, by the backwards form of the Triangle Inequality,

$$|z - w| \geq |z - \alpha| - |w - \alpha| = |z - \alpha| - R_1, \quad \text{for } w \in C_1.$$

It follows from the Estimation Theorem that

$$\begin{aligned} |I_n| &\leq \frac{1}{2\pi} \left( \frac{R_1^n}{|z - \alpha|^n} \times \frac{K}{|z - \alpha| - R_1} \right) 2\pi R_1 \\ &= \frac{KR_1}{|z - \alpha| - R_1} \left( \frac{R_1}{|z - \alpha|} \right)^n. \end{aligned}$$

Since  $|z - \alpha| > R_1$  (see Figure 2.10), the right-hand side tends to 0 as  $n \rightarrow \infty$ , so  $I_n$  tends to 0 as  $n \rightarrow \infty$ .

It follows from equation (2.6) that, on letting  $n \rightarrow \infty$ , we have

$$\begin{aligned} & -\frac{1}{2\pi i} \int_{C_1} \frac{f(w)}{w - z} dw \\ &= \frac{1}{2\pi i} \int_{C_1} \frac{f(w)}{z - \alpha} dw + \frac{1}{2\pi i} \int_{C_1} \frac{w - \alpha}{(z - \alpha)^2} f(w) dw \\ & \quad + \frac{1}{2\pi i} \int_{C_1} \frac{(w - \alpha)^2}{(z - \alpha)^3} f(w) dw + \cdots \\ &= \left( \frac{1}{2\pi i} \int_{C_1} f(w) dw \right) \frac{1}{z - \alpha} + \left( \frac{1}{2\pi i} \int_{C_1} (w - \alpha) f(w) dw \right) \frac{1}{(z - \alpha)^2} \\ & \quad + \left( \frac{1}{2\pi i} \int_{C_1} (w - \alpha)^2 f(w) dw \right) \frac{1}{(z - \alpha)^3} + \cdots \\ &= \sum_{n=1}^{\infty} b_n (z - \alpha)^{-n}, \end{aligned}$$

where

$$b_n = \frac{1}{2\pi i} \int_{C_1} (w - \alpha)^{n-1} f(w) dw, \quad \text{for } n = 1, 2, \dots$$

Since  $b_n$  is equal to  $a_{-n}$  in equation (2.5), we obtain the required singular part.

5. Finally, we establish the uniqueness property. Suppose that

$$f(z) = \sum_{n=-\infty}^{\infty} b_n (z - \alpha)^n, \quad \text{for } z \in A.$$

We wish to show that  $a_m = b_m$ , for all  $m \in \mathbb{Z}$ . Now, for each  $m \in \mathbb{Z}$ ,

$$\begin{aligned} \frac{f(z)}{(z-\alpha)^{m+1}} &= \left( \sum_{n=-\infty}^{m-1} b_n(z-\alpha)^{n-m-1} \right) \\ &\quad + \frac{b_m}{z-\alpha} + \left( \sum_{n=m+1}^{\infty} b_n(z-\alpha)^{n-m-1} \right) \\ &= f_1(z) + b_m/(z-\alpha) + f_2(z), \end{aligned} \quad (2.7)$$

say. But the function

$$F_2(z) = \sum_{n=m+1}^{\infty} \frac{b_n}{n-m} (z-\alpha)^{n-m}$$

is a primitive of  $f_2$  on the region  $\{z : |z-\alpha| < s\}$ , as you can check by differentiating term by term (which is permitted inside the disc of convergence by the Differentiation Rule for power series, Theorem 2.3 of Unit B3). It follows from the Closed Contour Theorem (Theorem 3.4 of Unit B1) that

$$\int_C f_2(z) dz = 0,$$

where  $C$  is any circle with centre  $\alpha$  lying in  $A$ .

Similarly, you can check that

$$F_1(z) = \sum_{n=-\infty}^{m-1} \frac{b_n}{n-m} (z-\alpha)^{n-m}$$

is a primitive of  $f_1$  on the region  $\{z : |z-\alpha| > r\}$ . Therefore, by the Closed Contour Theorem,

$$\int_C f_1(z) dz = 0.$$

Thus, by equation (2.7),

$$\begin{aligned} \int_C \frac{f(z)}{(z-\alpha)^{m+1}} dz &= \int_C f_1(z) dz + \int_C \frac{b_m}{z-\alpha} dz + \int_C f_2(z) dz \\ &= \int_C \frac{b_m}{z-\alpha} dz. \end{aligned}$$

But we know from Cauchy's Integral Formula or the method of parametrisation that

$$\frac{1}{2\pi i} \int_C \frac{b_m}{z-\alpha} dz = b_m.$$

Hence

$$a_m = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-\alpha)^{m+1}} dz = b_m,$$

so  $a_m = b_m$ , for all  $m \in \mathbb{Z}$ , as required. ■

We now prove Theorem 2.2.

**Theorem 2.2**

Let  $f$  be a function that has a singularity at the point  $\alpha$ , and suppose that the Laurent series about  $\alpha$  for  $f$  is

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z - \alpha)^n.$$

Then

- (a)  $f$  has a removable singularity at  $\alpha$  if and only if  
 $a_n = 0$  for all  $n < 0$
- (b)  $f$  has a pole of order  $k \in \mathbb{N}$  at  $\alpha$  if and only if  
 $a_{-k} \neq 0$  and  $a_n = 0$  for all  $n < -k$
- (c)  $f$  has an essential singularity at  $\alpha$  if and only if  
 $a_n \neq 0$  for infinitely many  $n < 0$ .

**Proof**

- (a) If  $f$  has a removable singularity at  $\alpha$ , then

$$f(z) = g(z), \quad \text{for } 0 < |z - \alpha| < r,$$

where  $r > 0$  and  $g$  is analytic on the disc  $\{z : |z - \alpha| < r\}$ . If the Taylor series about  $\alpha$  for  $g$  is

$$g(z) = \sum_{n=0}^{\infty} b_n(z - \alpha)^n, \quad \text{for } |z - \alpha| < r, \quad (2.8)$$

then

$$f(z) = \sum_{n=0}^{\infty} b_n(z - \alpha)^n, \quad \text{for } 0 < |z - \alpha| < r. \quad (2.9)$$

This last equation gives the Laurent series about  $\alpha$  for  $f$ , and hence  $a_n = 0$  for all  $n < 0$ , by uniqueness of Laurent series.

Conversely, if the Laurent series about  $\alpha$  for  $f$  satisfies  $a_n = 0$  for all  $n < 0$ , then it is of the form in equation (2.9). Thus  $f(z) = g(z)$ , for  $0 < |z - \alpha| < r$ , where  $g$  is defined by equation (2.8). It follows that  $f$  has a removable singularity at  $\alpha$ .

- (b) If  $f$  has a pole of order  $k$  at  $\alpha$ , then

$$f(z) = g(z)/(z - \alpha)^k, \quad \text{for } 0 < |z - \alpha| < r,$$

where  $r > 0$  and  $g$  is analytic on the disc  $\{z : |z - \alpha| < r\}$ , with  $g(\alpha) \neq 0$ .

If the Taylor series about  $\alpha$  for  $g$  is

$$g(z) = \sum_{n=0}^{\infty} b_n(z - \alpha)^n, \quad \text{for } |z - \alpha| < r, \quad (2.10)$$

then  $b_0 = g(\alpha) \neq 0$  and

$$f(z) = \frac{b_0}{(z - \alpha)^k} + \frac{b_1}{(z - \alpha)^{k-1}} + \cdots, \quad (2.11)$$

for  $0 < |z - \alpha| < r$ . This last equation gives the Laurent series about  $\alpha$  for  $f$ , and hence  $a_{-k} = b_0 \neq 0$  and  $a_n = 0$  for all  $n < -k$ .

Conversely, if the Laurent series about  $\alpha$  for  $f$  satisfies  $a_{-k} \neq 0$  and  $a_n = 0$  for all  $n < -k$ , then it is of the form in equation (2.11). Thus

$$f(z) = g(z)/(z - \alpha)^k, \quad \text{for } 0 < |z - \alpha| < r,$$

where  $g$  is defined by equation (2.10). It follows that  $f$  has a pole of order  $k$  at  $\alpha$ .

- (c) If  $f$  has an essential singularity at  $\alpha$ , then it has neither a removable singularity nor a pole at  $\alpha$ , and thus neither of the conditions on  $a_n$  in parts (a) and (b) applies. It follows that  $a_n \neq 0$  for infinitely many  $n < 0$ .

Conversely, if the Laurent series about  $\alpha$  for  $f$  satisfies  $a_n \neq 0$  for infinitely many  $n < 0$ , then  $f$  has neither a removable singularity nor a pole at  $\alpha$ , by parts (a) and (b). It follows that  $f$  has an essential singularity at  $\alpha$ . ■

## Further exercises

### Exercise 2.11

Find the annulus of convergence of each of the following extended power series.

- (a)  $\frac{i}{z} + 1 - z + z^2 - z^3 + \dots$   
 (b)  $1 + \frac{1}{z} - \frac{1}{z^2} + \frac{1}{z^3} - \dots$

### Exercise 2.12

Determine the Laurent series about 0 for the function

$$f(z) = \left( \frac{1}{z} - \frac{1}{z^2} \right) \sin z.$$

State a punctured disc on which the function  $f$  is represented by this Laurent series.

### Exercise 2.13

Find the Laurent series for the function

$$f(z) = \frac{1}{z(z-4)}$$

- (a) about 0 on  $\{z : 0 < |z| < 4\}$   
 (b) about 0 on  $\{z : |z| > 4\}$   
 (c) about 4 on  $\{z : 0 < |z - 4| < 4\}$ .

## Exercise 2.14

Find the Laurent series for the function

$$f(z) = \frac{1}{(z-1)(z-3)}$$

- (a) about 0 on  $\{z : |z| < 1\}$
- (b) about 0 on  $\{z : |z| > 3\}$
- (c) about 1 on  $\{z : 0 < |z-1| < 2\}$ .

## Exercise 2.15

Show that each of the following functions has an essential singularity at 0.

- (a)  $f(z) = \cos \frac{1}{z}$
- (b)  $f(z) = z \sinh \frac{1}{z}$

## Fourier series

Suppose that  $f$  is an analytic function with Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n, \quad \text{for } 1/r < |z| < r,$$

where  $r > 1$ . The Laurent series converges on the circle  $|z| = 1$ , so we can define a *real* function  $g$  by  $g(\theta) = f(e^{i\theta})$ , where  $\theta \in \mathbb{R}$ . Using the formula for  $f$  we see that

$$g(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta}. \quad (2.12)$$

A series of this form is called a complex **Fourier series** (where Fourier is pronounced ‘foo-ree-ay’, or similar). The function  $g$  is periodic with period  $2\pi$ , meaning that  $g(\theta + 2\pi) = g(\theta)$ , for all  $\theta \in \mathbb{R}$ .

Fourier series are named after the French mathematician and physicist Joseph Fourier (1768–1830). In 1807 he made the remarkable discovery that any well-behaved real periodic function  $g$  can be expressed in the form

$$g(\theta) = a_0 + \sum_{n=1}^{\infty} (c_n \cos n\theta + d_n \sin n\theta), \quad (2.13)$$

where  $a_0$ ,  $c_n$  and  $d_n$  are real numbers. If we define

$$a_n = \frac{c_n - id_n}{2} \quad \text{and} \quad a_{-n} = \frac{c_n + id_n}{2}, \quad \text{for } n = 1, 2, \dots,$$

then a short computation shows that the two equations (2.12) and (2.13) are equivalent.



Joseph Fourier

Fourier series are of fundamental importance in a huge range of physical problems involving oscillations and waves, because they provide a means for modelling waves mathematically. Fourier himself used these series to offer the first general solution of the **heat equation**, a differential equation that models the temperature at each point of a body as a function of time.

## 3 Behaviour near a singularity

After working through this section, you should be able to:

- characterise a removable singularity of a function  $f$  in terms of the behaviour of  $f$  near that singularity
- characterise a pole of a function  $f$  in terms of the behaviour of  $f$  near that pole
- state the Casorati–Weierstrass Theorem concerning the behaviour of a function near an essential singularity.

In this section we investigate the behaviour of analytic functions near each of the three types of singularity.

### 3.1 Removable singularities

Recall that a function  $f$  has a *removable singularity* at  $\alpha$  if  $f$  is analytic on a punctured open disc  $\{z : 0 < |z - \alpha| < r\}$ , but not at  $\alpha$  itself, and there is a function  $g$  that is analytic on the open disc  $\{z : |z - \alpha| < r\}$  such that

$$f(z) = g(z), \quad \text{for } 0 < |z - \alpha| < r. \quad (3.1)$$

The following theorem gives three equivalent conditions for  $f$  to have a removable singularity at  $\alpha$ .

#### Theorem 3.1

Let  $f$  be a function that has a singularity at the point  $\alpha$ . Then the following statements are equivalent:

- $f$  has a removable singularity at  $\alpha$ .
- $\lim_{z \rightarrow \alpha} f(z)$  exists.
- $f$  is bounded on  $\{z : 0 < |z - \alpha| < r\}$ , for some  $r > 0$ .
- $\lim_{z \rightarrow \alpha} (z - \alpha)f(z) = 0$ .

**Proof** To prove the equivalence of the four statements, it is sufficient to show that

$$(a) \implies (b) \implies (c) \implies (d) \implies (a).$$

We do this in four steps.

**(a)  $\implies$  (b)** Let  $g$  be the function that is analytic on  $\{z : |z - \alpha| < r\}$  and that satisfies equation (3.1). Since  $g$  is analytic, and hence continuous, at  $\alpha$  and  $f(z) = g(z)$ , for  $0 < |z - \alpha| < r$ , we have

$$\lim_{z \rightarrow \alpha} f(z) = \lim_{z \rightarrow \alpha} g(z) = g(\alpha).$$

**(b)  $\implies$  (c)** First we use condition (b) to extend the domain of  $f$  to include the point  $\alpha$  (if it is not defined there already) by using the formula

$$f(\alpha) = \lim_{z \rightarrow \alpha} f(z).$$

If  $f$  is already defined at  $\alpha$ , then we use this last equality to redefine  $f$  at  $\alpha$ . By the definition of  $f$  at  $\alpha$ , we see that  $f$  is continuous at  $\alpha$ . We now apply the  $\varepsilon$ - $\delta$  definition of continuity at  $\alpha$  with  $\varepsilon = 1$  to see that there is a positive number  $r$  (which has the role of  $\delta$ ) such that

$$|f(z) - f(\alpha)| < 1, \quad \text{for } |z - \alpha| < r.$$

So if  $z$  belongs to the open disc  $D = \{z : |z - \alpha| < r\}$ , then, by the Triangle Inequality,

$$|f(z)| \leq |f(z) - f(\alpha)| + |f(\alpha)| < 1 + |f(\alpha)|.$$

It follows that  $f$  is bounded on  $D$ , so it is bounded on the punctured disc  $D - \{\alpha\}$ .

**(c)  $\implies$  (d)** Suppose that  $|f(z)| \leq K$ , for  $0 < |z - \alpha| < r$ . Then

$$|(z - \alpha)f(z)| \leq K|z - \alpha|, \quad \text{for } 0 < |z - \alpha| < r,$$

so

$$\lim_{z \rightarrow \alpha} (z - \alpha)f(z) = 0.$$

**(d)  $\implies$  (a)** Choose  $r > 0$  such that the function  $f$  is analytic on  $\{z : 0 < |z - \alpha| < r\}$ . Consider the function

$$g(z) = \begin{cases} (z - \alpha)^2 f(z), & 0 < |z - \alpha| < r, \\ 0, & z = \alpha, \end{cases}$$

which is clearly analytic on  $\{z : 0 < |z - \alpha| < r\}$ . Then, for  $0 < |z - \alpha| < r$ ,

$$\frac{g(z) - g(\alpha)}{z - \alpha} = \frac{(z - \alpha)^2 f(z)}{z - \alpha} = (z - \alpha)f(z) \rightarrow 0 \text{ as } z \rightarrow \alpha,$$

by condition (d). Hence  $g$  is differentiable at  $\alpha$ , and  $g'(\alpha) = 0$ . Therefore  $g$  is analytic on  $\{z : |z - \alpha| < r\}$ , so it can be represented by its Taylor series about  $\alpha$  on this open disc. Since  $g(\alpha) = g'(\alpha) = 0$ , this Taylor series takes the form

$$g(z) = \sum_{m=2}^{\infty} a_m (z - \alpha)^m, \quad \text{for } |z - \alpha| < r.$$

Hence, by the definition of  $g$  and by setting  $m = n + 2$ , we see that

$$\begin{aligned} f(z) &= \sum_{m=2}^{\infty} a_m (z - \alpha)^{m-2} \\ &= \sum_{n=0}^{\infty} a_{n+2} (z - \alpha)^n, \quad \text{for } 0 < |z - \alpha| < r. \end{aligned}$$

Thus, by Theorem 2.2(a),  $f$  has a removable singularity at  $\alpha$ . ■

### Exercise 3.1

Verify condition (d) of Theorem 3.1 for each of the following functions.

(a)  $f(z) = \frac{\sin^2 z}{z^2}, \quad \alpha = 0$       (b)  $f(z) = \frac{3z}{\tan 3z}, \quad \alpha = 0$

(c)  $f(z) = \frac{z^2 + 3iz - 2}{z^2 + 4}, \quad \alpha = -2i$

## 3.2 Poles

Recall that a function  $f$  has a *pole of order  $k$*  at  $\alpha$  if  $f$  is analytic on a punctured open disc  $\{z : 0 < |z - \alpha| < r\}$ , but not at  $\alpha$  itself, and there is a function  $g$  that is analytic on the open disc  $\{z : |z - \alpha| < r\}$  such that  $g(\alpha) \neq 0$  and

$$f(z) = \frac{g(z)}{(z - \alpha)^k}, \quad \text{for } 0 < |z - \alpha| < r. \quad (3.2)$$

The following theorem gives two equivalent conditions for  $f$  to have a pole of order  $k$  at  $\alpha$ .

### Theorem 3.2

Let  $f$  be a function that has a singularity at the point  $\alpha$ , and let  $k \in \mathbb{N}$ . Then the following statements are equivalent:

- (a)  $f$  has a pole of order  $k$  at  $\alpha$ .
- (b)  $\lim_{z \rightarrow \alpha} (z - \alpha)^k f(z)$  exists, and is non-zero.
- (c)  $1/f$  has a removable singularity at  $\alpha$  which, when removed, gives rise to a zero of order  $k$  at  $\alpha$ .

**Proof** To prove the equivalence of the three statements, it is sufficient to show that

$$(a) \implies (b) \implies (c) \implies (a).$$

We do this in three steps.

**(a)  $\implies$  (b)** Let  $g$  be the function that is analytic on  $\{z : |z - \alpha| < r\}$ , with  $g(\alpha) \neq 0$ , and that satisfies equation (3.2). Since  $g$  is analytic, and hence continuous, at  $\alpha$ , and  $f(z) = g(z)/(z - \alpha)^k$ , for  $0 < |z - \alpha| < r$ , we have

$$\lim_{z \rightarrow \alpha} (z - \alpha)^k f(z) = \lim_{z \rightarrow \alpha} g(z) = g(\alpha),$$

which is non-zero.

**(b)  $\implies$  (c)** Let  $g(z) = (z - \alpha)^k f(z)$  (a different  $g$  to that of the previous step). Then, by Theorem 3.1 ((b)  $\implies$  (a)),  $g$  has a removable singularity at  $\alpha$ , which we can remove by defining

$$g(\alpha) = \lim_{z \rightarrow \alpha} (z - \alpha)^k f(z)$$

(if  $g$  is already defined at  $\alpha$ , then we redefine  $g(\alpha)$  using this formula). Furthermore,  $g$  is now continuous at  $\alpha$ , and  $g(\alpha) \neq 0$  by condition (b), so there exists  $r > 0$  such that  $g(z) \neq 0$ , for  $|z - \alpha| < r$ .

It follows that

$$\frac{1}{f(z)} = (z - \alpha)^k \frac{1}{g(z)}, \quad \text{for } |z - \alpha| < r,$$

and  $1/g$  is analytic at  $\alpha$ , with  $1/g(\alpha) \neq 0$ .

Thus  $1/f$  has a zero of order  $k$  at  $\alpha$ , by Theorem 5.1 of Unit B3.

**(c)  $\implies$  (a)** By condition (c),

$$\frac{1}{f(z)} = (z - \alpha)^k g(z),$$

where  $g$  is analytic at  $\alpha$  with  $g(\alpha) \neq 0$  (again, a different  $g$  to those of the previous steps). Thus there exists  $r > 0$  such that  $g(z) \neq 0$ , for  $|z - \alpha| < r$ .

It follows that

$$f(z) = \frac{1/g(z)}{(z - \alpha)^k}, \quad \text{for } 0 < |z - \alpha| < r,$$

and the function  $1/g$  is analytic on  $\{z : |z - \alpha| < r\}$ , with  $1/g(\alpha) \neq 0$ .

Thus  $f$  has a pole of order  $k$  at  $\alpha$ . ■

### Corollary

Let  $f$  be a function that has a singularity at the point  $\alpha$ . Then  $f$  has a pole at  $\alpha$  if and only if  $f(z) \rightarrow \infty$  as  $z \rightarrow \alpha$ .

Some texts take the condition ' $f(z) \rightarrow \infty$  as  $z \rightarrow \alpha$ ' as their *definition* of a pole.

**Proof** First assume that  $f$  has a pole at  $\alpha$ , of order  $k$ , say. Then, by condition (b) of Theorem 3.2,

$$\lim_{z \rightarrow \alpha} (z - \alpha)^k f(z) = \lambda, \quad \text{where } \lambda \neq 0.$$

It follows that

$$\frac{1}{f(z)} = \frac{(z - \alpha)^k}{(z - \alpha)^k f(z)} \rightarrow \frac{0}{\lambda} = 0 \text{ as } z \rightarrow \alpha.$$

Thus  $f(z) \rightarrow \infty$  as  $z \rightarrow \alpha$ , by the Reciprocal Rule (Theorem 1.1).

Conversely, suppose that  $f(z) \rightarrow \infty$  as  $z \rightarrow \alpha$ . Then, by the Reciprocal Rule,

$$\frac{1}{f(z)} \rightarrow 0 \text{ as } z \rightarrow \alpha.$$

Thus the function  $1/f$  has a removable singularity at  $\alpha$ , by Theorem 3.1 ((b)  $\implies$  (a)), which, when removed, gives rise to a zero of  $1/f$ . Hence  $f$  has a pole at  $\alpha$ , by Theorem 3.2 ((c)  $\implies$  (a)). ■

The next exercise uses the functions from Exercise 1.7, and you may find it helpful to refer to your solutions to that exercise.

### Exercise 3.2

Verify condition (b) of Theorem 3.2 for each of the following functions.

$$(a) \ f(z) = \frac{z+2}{z^4(z^2-4)^3} \quad (b) \ f(z) = \frac{z}{\sin^3 z}$$

### Exercise 3.3

Suppose that the function  $f$  has a singularity at 0. In each of the following cases, classify this singularity.

- (a)  $f(z) = g(z)/z$ , where  $g$  is entire and  $g(0) \neq 0$ .
- (b)  $f(z) = g(z)/z$ , where  $g$  is entire and  $g(0) = 0$ .
- (c)  $f(z) = 1/g(z)$ , where  $g$  is entire and has a zero of order 2 at 0.

### Exercise 3.4

Suppose that the function  $f$  has a pole of order  $m$  at  $\alpha$ , and the function  $g$  has a pole of order  $n$  at  $\alpha$ . Classify the singularities of  $f+g$  and  $fg$  at  $\alpha$  in each of the following cases.

- (a)  $m = 5, n = 3$       (b)  $m = n = 4$

### 3.3 Essential singularities

Recall that a singularity is an *essential singularity* if it is neither a removable singularity nor a pole. This rather negative-sounding definition might suggest that essential singularities are uninteresting. In fact, the opposite is true.

We have seen that if  $\alpha$  is a removable singularity of  $f$ , then  $f(z)$  tends to a finite limit as  $z$  tends to  $\alpha$ . We have also seen that if  $\alpha$  is a pole of  $f$ , then  $f(z)$  tends to  $\infty$  as  $z$  tends to  $\alpha$ . Let us now explore what happens to  $f(z)$  as  $z$  tends to  $\alpha$  if  $\alpha$  is an essential singularity.

We will see that, in a sense, the function goes haywire! A celebrated result due to Émile Picard, whom you met in Unit B2, states that in any punctured open disc with centre  $\alpha$ , however small,  $f$  takes all values in  $\mathbb{C}$ , with at most one exception. This result is known as *Picard's Big Theorem* (or Picard's Great Theorem) and it is possible to deduce Picard's Little Theorem (mentioned in Subsection 2.3 of Unit B2) from this stronger result.

Proving Picard's Big Theorem is beyond the scope of this module, but we can prove a related result which is itself surprising. It is known as the Casorati–Weierstrass Theorem (Casorati is pronounced ‘cas-oh-rah-tee’), and asserts that as  $z$  varies over any punctured disc with centre  $\alpha$ , the values  $f(z)$  come arbitrarily close to any given complex number  $w$ . We now state this result formally.

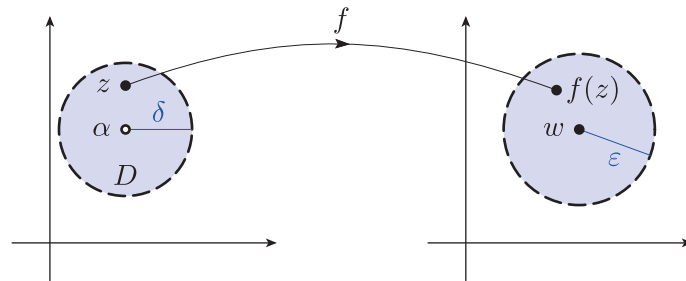
#### Theorem 3.3 Casorati–Weierstrass Theorem

Suppose that a function  $f$  has an essential singularity at  $\alpha$ .

Let  $D$  be any punctured open disc  $\{z : 0 < |z - \alpha| < \delta\}$  centred at  $\alpha$ , and let  $w$  be any complex number. Then, for any positive number  $\varepsilon$ ,

there exists  $z \in D$  such that  $|f(z) - w| < \varepsilon$ .

The theorem is illustrated in Figure 3.1.



**Figure 3.1** A point  $z$  in  $D = \{z : 0 < |z - \alpha| < \delta\}$  mapped by  $f$  to within a distance  $\varepsilon$  of  $w$

**Proof** We assume that the result is false and seek a contradiction. Then there exist  $w \in \mathbb{C}$  and positive real numbers  $\varepsilon$  and  $\delta$  such that the

function  $f$  is analytic on the punctured open disc  $D = \{z : 0 < |z - \alpha| < \delta\}$  and the last line of the theorem does not hold; that is,

$$|f(z) - w| \geq \varepsilon, \quad \text{for } z \in D.$$

We will derive a contradiction from this statement.

Since  $f(z) - w \neq 0$ , for  $z \in D$ , the function

$$g(z) = \frac{1}{f(z) - w} \quad (z \in D)$$

is analytic. Moreover,

$$|g(z)| = \frac{1}{|f(z) - w|} \leq 1/\varepsilon, \quad \text{for } z \in D,$$

so, by Theorem 3.1 ((c)  $\implies$  (a)),  $g$  has a removable singularity at  $\alpha$ . Thus, by defining  $g(\alpha)$  appropriately, we can remove the singularity so that  $g$  is analytic on the disc  $\{z : |z - \alpha| < \delta\}$ .

Now  $g(z) \neq 0$ , for  $z \in D$ , so

$$f(z) = w + \frac{1}{g(z)}, \quad \text{for } z \in D.$$

If  $g(\alpha) \neq 0$ , then  $f$  would have a removable singularity at  $\alpha$ , which could be removed by letting  $f(\alpha) = w + 1/g(\alpha)$ .

If  $g(\alpha) = 0$ , then  $g(z) \rightarrow 0$  as  $z \rightarrow \alpha$ , so  $1/g(z) \rightarrow \infty$  as  $z \rightarrow \alpha$ , by the Reciprocal Rule (Theorem 1.1). Hence

$$f(z) = w + \frac{1}{g(z)} \rightarrow \infty \text{ as } z \rightarrow \alpha,$$

so  $f$  has a pole at  $\alpha$ , by the corollary to Theorem 3.2.

Therefore  $f$  does not have an essential singularity at  $\alpha$ , which is a contradiction, and the theorem is thereby established. ■

### Exercise 3.5

Prove that there exists a complex number  $z$  such that

$$|z| < 10^{-3} \quad \text{and} \quad |e^{1/z} - 1000i| < 10^{-6}.$$

## Further exercises

### Exercise 3.6

Use Theorem 3.1 and the corollary to Theorem 3.2 to prove that if the function  $f$  has an essential singularity at a point  $\alpha$ , then the function

$$g(z) = (f(z))^2$$

has an essential singularity at  $\alpha$ .

(You may assume that if  $g(z) \rightarrow \infty$  as  $z \rightarrow \alpha$ , then  $f(z) \rightarrow \infty$  as  $z \rightarrow \alpha$ .)



Felice Casorati

### Origin of the Casorati–Weierstrass Theorem

The Casorati–Weierstrass Theorem appears to have been discovered independently by *three* different mathematicians.

Felice Casorati (1835–1890) was an Italian mathematician who published a text *Teorica delle funzioni di variabili complesse* on complex analysis in 1868, which included the theorem. In preparing the text, Casorati visited Berlin to meet Weierstrass and others, and he was greatly influenced by this leading group.

Weierstrass did eventually publish the theorem himself, in 1876. He wrote, of a function  $f$  with an essential singularity  $c$ , that

in an infinitely small neighbourhood of the point  $c$  the function  $f(x)$  behaves in such a discontinuous way that it can come arbitrarily close to any arbitrarily given value, for  $x = c$  however, it has no determinate value.

(Bottazzini and Gray, 2013, p. 436)

Much to Casorati's irritation, Weierstrass did not acknowledge that Casorati had also discovered this theorem. As Casorati wrote, Weierstrass

had certainly found the theorem by himself, but to whom I had sent my *Teorica* as a present in 1868.

(Neuenschwander, 1978, p. 7, cited in Bottazzini and Gray, 2013, p. 436)

Independently of Casorati and Weierstrass, the theorem was also discovered by the Polish–Russian mathematician Yulian Vasilievich Sokhotskii (1842–1927) and published in his masters dissertation in 1868. He wrote:

If a given function  $f(z)$  becomes infinite of infinite order at some point, then the function  $f(z)$  must take all possible values at that point.

(Mitrinović and Kečkić, 1993, p. 94, cited in Bottazzini and Gray, 2013, p. 214)

This quotation sounds more like the statement of Picard's Big Theorem than the Casorati–Weierstrass Theorem; however, it is clear from the proof that Sokhotskii intended only to prove the theorem also established by Casorati and Weierstrass.

In Russian literature the Casorati–Weierstrass Theorem is often called Sokhotskii's Theorem, and in some modern texts the theorem is referred to as the Casorati–Sokhotskii–Weierstrass Theorem.

## 4 Evaluating integrals using Laurent series

After working through this section, you should be able to:

- use Laurent's Theorem to evaluate integrals
- define the *residue* of a function  $f$  with a singularity at  $\alpha$ .

### 4.1 Integration and residues

Laurent's Theorem tells us that if  $f$  is a function that is analytic on the punctured disc  $D = \{z : 0 < |z - \alpha| < r\}$ , then the coefficient of  $(z - \alpha)^n$  in the Laurent series about  $\alpha$  for  $f$  can be expressed in the form

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(w)}{(w - \alpha)^{n+1}} dw, \quad \text{for } n \in \mathbb{Z},$$

where  $C$  is any circle lying in  $D$  with centre  $\alpha$ .

As with Taylor series, this integral expression is of little use for calculating the coefficients  $a_n$ : it is essentially of theoretical value only. In practice we usually reverse the process, calculating the coefficients by other means and then using them to evaluate integrals by the formula

$$\int_C \frac{f(w)}{(w - \alpha)^{n+1}} dw = 2\pi i a_n. \quad (4.1)$$

Here is an example.

#### Example 4.1

Use the Laurent series about 0 for the function  $f(z) = \cos(1/z)$  to evaluate the following integrals, where  $C$  is any circle with centre 0.

$$(a) \int_C w^{-5} \cos(1/w) dw \quad (b) \int_C w^5 \cos(1/w) dw$$

#### Solution

The function  $f(z) = \cos(1/z)$  is analytic on the set  $\mathbb{C} - \{0\}$ , which contains any circle  $C$  with centre 0, and

$$\cos(1/z) = 1 - \frac{1}{2! z^2} + \frac{1}{4! z^4} - \frac{1}{6! z^6} + \cdots, \quad \text{for } z \in \mathbb{C} - \{0\}.$$

Hence, using equation (4.1) with  $\alpha = 0$ ,

$$(a) \int_C w^{-5} \cos(1/w) dw = \int_C \frac{f(w)}{w^5} dw = 2\pi i a_4 = 0$$

$$(b) \int_C w^5 \cos(1/w) dw = \int_C \frac{f(w)}{w^{-5}} dw = 2\pi i a_{-6} = -\frac{\pi i}{360}.$$

## Exercise 4.1

Evaluate the following integrals, where  $C$  is any circle with centre 0.

$$(a) \int_C w^{-5} \sinh(1/w) dw \quad (b) \int_C w^4 \sinh(1/w) dw$$

Note that if we take  $n = -1$  in equation (4.1), then we obtain

$$\int_C f(w) dw = 2\pi i a_{-1}. \quad (4.2)$$

It follows that we can evaluate the integral on the left by calculating the appropriate coefficient  $a_{-1}$ . This turns out to be exceedingly useful in practice and, because of its importance, the coefficient  $a_{-1}$  is given a special name.

## Definition

Let  $f$  be a function that is analytic on a punctured disc with centre  $\alpha$ . The **residue of  $f$  at  $\alpha$**  is the coefficient  $a_{-1}$  of  $(z - \alpha)^{-1}$  in the Laurent series about  $\alpha$  for  $f$ . It is denoted by  $\text{Res}(f, \alpha)$ .

Residues will feature heavily in Unit C1 when we study the Residue Theorem.

We can now write equation (4.2) as

$$\int_C f(z) dz = 2\pi i \text{Res}(f, \alpha), \quad (4.3)$$

where  $C$  is any circle in the punctured disc with centre  $\alpha$  (and where we have changed the variable of integration from  $w$  to  $z$ ).

The following example illustrates the use of this result.

## Example 4.2

Evaluate each of the following integrals.

- (a)  $\int_C \frac{1}{z - 3i} dz$ , where  $C = \{z : |z - 3i| = 1\}$   
 (b)  $\int_C \sin(i/z) dz$ , where  $C = \{z : |z| = 5\}$   
 (c)  $\int_C \frac{\sin z}{z} dz$ , where  $C = \{z : |z| = 1\}$

## Solution

- (a) The function  $f(z) = 1/(z - 3i)$  has a simple pole at  $3i$ , and the Laurent series about  $3i$  for  $f$  is simply

$$\frac{1}{z - 3i} = \frac{1}{z - 3i} + 0 + 0(z - 3i) + \cdots, \quad \text{for } z \in \mathbb{C} - \{3i\},$$

so  $\text{Res}(f, 3i) = 1$ . Since  $C$  has centre  $3i$ , we obtain, by equation (4.3),

$$\int_C \frac{1}{z - 3i} dz = 2\pi i \text{Res}(f, 3i) = 2\pi i.$$

- (b) The function  $f(z) = \sin(i/z)$  has an essential singularity at 0, and the Laurent series about 0 for  $f$  is

$$\sin\left(\frac{i}{z}\right) = \frac{i}{z} - \frac{i^3}{3!z^3} + \frac{i^5}{5!z^5} - \cdots, \quad \text{for } z \in \mathbb{C} - \{0\},$$

so  $\text{Res}(f, 0) = i$ . Since  $C$  has centre 0, we obtain, by equation (4.3),

$$\int_C \sin(i/z) dz = 2\pi i \text{Res}(f, 0) = -2\pi.$$

- (c) The function  $f(z) = (\sin z)/z$  has a removable singularity at 0, and the Laurent series about 0 for  $f$  is

$$\frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots, \quad \text{for } z \in \mathbb{C} - \{0\},$$

so  $\text{Res}(f, 0) = 0$ . Since  $C$  has centre 0, we obtain, by equation (4.3),

$$\int_C \frac{\sin z}{z} dz = 2\pi i \text{Res}(f, 0) = 0.$$

### Remarks

1. Note that the result of part (a) could also have been obtained by using Cauchy's Integral Formula, or by parametrisation, but this is not true for the result of part (b).
2. In part (c), if we define  $f$  at 0 by  $f(0) = 1$ , then  $f$  becomes an entire function, so, by Cauchy's Theorem,

$$\int_C f(z) dz = 0,$$

in agreement with the solution to part (c).

### Exercise 4.2

Evaluate each of the following integrals.

- (a)  $\int_C \frac{1}{(z+i)^2} dz$ , where  $C = \{z : |z+i| = 2\}$ .
- (b)  $\int_C \frac{\sin 2z}{z^4} dz$ , where  $C = \{z : |z| = 5\}$ .

In the next unit we will meet several methods for calculating residues, and we will discuss other types of integral that can be evaluated by means of residues.

## 4.2 Revision of contour integration

We conclude this book with a revision exercise where you can apply the various techniques that you have learned. It is a ‘no-holds-barred’ exercise, so you can use any of the methods in the book: direct calculation by parametrisation; the Fundamental Theorem of Calculus, Cauchy’s Theorem or Cauchy’s Formulas; Integration by Parts; calculation of an appropriate residue. Often more than one method will work; the challenge lies in determining an appropriate or quick method for each question.

### Exercise 4.3

Evaluate as many of the following integrals as you have time for, using whichever method seems most appropriate. For each integral, you should quote any results that you use, and check that their hypotheses are satisfied.

- (a)  $\int_C e^z/z \, dz$ , where  $C = \{z : |z| = 3\}$ .
- (b)  $\int_C e^z/z \, dz$ , where  $C = \{z : |z - 1| = \frac{1}{2}\}$ .
- (c)  $\int_C \sec^2 z \, dz$ , where  $C = \{z : |z - \pi/2| = 1\}$ .
- (d)  $\int_C (\cosh z)/z^5 \, dz$ , where  $C = \{z : |z| = \pi\}$ .
- (e)  $\int_C \frac{z + \frac{1}{2} \sin 2z}{(z - \pi/4)^2} \, dz$ , where  $C = \{z : |z - \pi/4| = 1\}$ .
- (f)  $\int_C z \operatorname{cosec} z \, dz$ , where  $C = \{z : |z| = 1\}$ .
- (g)  $\int_C \exp(1/z^4) \, dz$ , where  $C = \{z : |z| = 2\}$ .
- (h)  $\int_C e^{1/z} \sin(1/z) \, dz$ , where  $C = \{z : |z| = 4\}$ .
- (i)  $\int_C \frac{e^z}{z^2(z-1)} \, dz$ , where  $C = \{z : |z| = \frac{1}{2}\}$ .
- (j)  $\int_C \frac{2 \sin \pi z}{z^2 - 1} \, dz$ , where  $C = \{z : |z| = 2\}$ .

# Solutions to exercises

## Solution to Exercise 1.1

(a) The function

$$f(z) = \frac{z + 2i}{(z - 3)^2(z^2 + 1)}$$

is analytic everywhere except at the zeros of the denominator. Thus the only possible singularities are at 3,  $i$  and  $-i$ . About each of these points we can find a punctured open disc on which  $f$  is analytic – for example:

about 3, the punctured disc  $\{z : 0 < |z - 3| < 3\}$

about  $i$ , the punctured disc  $\{z : 0 < |z - i| < 2\}$

about  $-i$ , the punctured disc  $\{z : 0 < |z + i| < 2\}$ .

Hence the singularities of  $f$  are at 3,  $i$ ,  $-i$ .

(b) The function

$$f(z) = \frac{3z - i}{z^3} \sin\left(\frac{1}{z + 1}\right)$$

is analytic everywhere except at 0 and  $-1$ . Thus the only possible singularities are at 0 and  $-1$ .

About each of these points we can find a punctured open disc on which the function  $f$  is analytic – for example:

about 0, the punctured disc  $\{z : 0 < |z| < 1\}$

about  $-1$ , the punctured disc  $\{z : 0 < |z + 1| < 1\}$ .

Hence the singularities of  $f$  are at 0,  $-1$ .

(c) The function

$$f(z) = \frac{4e^{-z}}{z^2 + 2iz - 1}$$

is analytic everywhere except at the zeros of  $z^2 + 2iz - 1 = (z + i)^2$ . Thus the only possible singularity is at  $-i$ . (Note that  $e^{-z}$  has no singularities.) About this point we can find a punctured open disc on which  $f$  is analytic – for example,

$$\{z : 0 < |z + i| < 100\}.$$

Hence the (only) singularity of  $f$  is at  $-i$ .

## Solution to Exercise 1.2

The function

$$f(z) = \frac{1}{\sin(1/z)}$$

is analytic everywhere except when  $z = 0$  and when  $\sin(1/z) = 0$ . Now,  $\sin(1/z) = 0$  if and only if  $1/z = 0, \pm\pi, \pm2\pi, \pm3\pi, \dots$ , so the only possible singularities are at 0 and

$$\pm 1/\pi, \pm 1/(2\pi), \pm 1/(3\pi), \dots$$

Although  $f$  is not defined at 0 (and thus is not analytic there), 0 is not a singularity of  $f$  because there is no punctured open disc about 0 on which  $f$  is analytic. In fact, *any* punctured disc  $\{z : 0 < |z| < r\}$  must contain infinitely many of the points  $\pm 1/\pi, \pm 1/(2\pi), \pm 1/(3\pi), \dots$ .

Consider now a non-zero integer  $k$ . Then  $f$  is analytic on the punctured disc

$$\{z : 0 < |z - 1/(k\pi)| < r_k\},$$

where  $r_k$  is the distance from  $1/(k\pi)$  to the nearer of  $1/((k+1)\pi)$  and  $1/((k-1)\pi)$ . Therefore the singularities of  $f$  are at the points

$$\pm 1/\pi, \pm 1/(2\pi), \pm 1/(3\pi), \dots$$

## Solution to Exercise 1.3

(a) Let  $A = \mathbb{C} - \{i\}$  and  $\alpha = i$ ; then  $\alpha$  is a limit point of  $A$ . Let  $f(z) = 1/(z - i)$ . Then

$$\lim_{z \rightarrow i} 1/f(z) = \lim_{z \rightarrow i} (z - i) = 0.$$

Hence, by the Reciprocal Rule,

$$1/(z - i) \rightarrow \infty \text{ as } z \rightarrow i.$$

(b) Let  $A = \mathbb{C} - \{0\}$  and  $\alpha = 0$ ; then  $\alpha$  is a limit point of  $A$ . Let  $f(z) = (\sin z)/z^2$ . Then

$$\begin{aligned} \lim_{z \rightarrow 0} 1/f(z) &= \lim_{z \rightarrow 0} z^2/\sin z \\ &= \lim_{z \rightarrow 0} z/\left(\frac{\sin z}{z}\right) \\ &= 0/1 = 0. \end{aligned}$$

Hence, by the Reciprocal Rule,

$$(\sin z)/z^2 \rightarrow \infty \text{ as } z \rightarrow 0.$$

(c) Let  $A = \mathbb{C} - \{-i, i\}$  and  $\alpha = i$ ; then  $\alpha$  is a limit point of  $A$ . Let  $f(z) = (z+i)/(z^2+1)^3$ . Then

$$\begin{aligned}\lim_{z \rightarrow i} 1/f(z) &= \lim_{z \rightarrow i} (z^2+1)^3/(z+i) \\ &= 0^3/(2i) = 0.\end{aligned}$$

Hence, by the Reciprocal Rule,

$$(z+i)/(z^2+1)^3 \rightarrow \infty \text{ as } z \rightarrow i.$$

### Solution to Exercise 1.4

Let  $f$  be a function with domain  $A$ , and let  $\alpha$  be a limit point of  $A$ .

Assume that  $f(z) \rightarrow \infty$  as  $z \rightarrow \alpha$ . Then, for each sequence  $(z_n)$  in  $A - \{\alpha\}$  such that  $z_n \rightarrow \alpha$ ,

$$f(z_n) \rightarrow \infty \text{ as } n \rightarrow \infty.$$

By the Reciprocal Rule for sequences,

$$1/f(z_n) \rightarrow 0 \text{ as } n \rightarrow \infty,$$

and thus

$$\lim_{z \rightarrow \alpha} 1/f(z) = 0.$$

Conversely, assume that  $\lim_{z \rightarrow \alpha} 1/f(z) = 0$ . Then, for each sequence  $(z_n)$  in  $A - \{\alpha\}$  such that  $z_n \rightarrow \alpha$ ,

$$1/f(z_n) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

By the Reciprocal Rule for sequences,

$$f(z_n) \rightarrow \infty \text{ as } n \rightarrow \infty,$$

and thus

$$f(z) \rightarrow \infty \text{ as } z \rightarrow \alpha.$$

### Solution to Exercise 1.5

(a) The function  $f(z) = (\sin^2 z)/z^2$  has a singularity at 0 and is analytic on  $\mathbb{C} - \{0\}$ . Also,

$$\begin{aligned}f(z) &= \frac{\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right)^2}{z^2} \\ &= \left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots\right)^2,\end{aligned}$$

for  $z \in \mathbb{C} - \{0\}$ . Let  $g$  be the function

$$g(z) = \left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots\right)^2,$$

which is entire, since it is given by the product of two identical power series, each of which determines an entire function. Then

$$f(z) = g(z), \quad \text{for } z \in \mathbb{C} - \{0\},$$

so  $f$  has a removable singularity at 0.

(b) The function  $f(z) = 3z \cot z$  has singularities at the points  $z = k\pi$ , for  $k \in \mathbb{Z}$ , where  $\cot z$  is not defined, and  $f$  is analytic on  $\mathbb{C} - \{k\pi : k \in \mathbb{Z}\}$ .

If  $k \neq 0$ , then

$$\lim_{z \rightarrow k\pi} \frac{1}{f(z)} = \frac{\sin z}{3z \cos z} = 0,$$

so  $f(z) \rightarrow \infty$  as  $z \rightarrow k\pi$ , by the Reciprocal Rule.

Therefore the singularities at  $k\pi$ , for  $k \neq 0$ , are not removable.

For the singularity at 0, observe that  $f$  is analytic on  $D = \{z : 0 < |z| < \pi/2\}$  and

$$\begin{aligned}f(z) &= \frac{3 \cos z}{(\sin z)/z} \\ &= \frac{3 \cos z}{1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots}, \quad \text{for } z \in D.\end{aligned}$$

Observe that

$$h(z) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots$$

is an entire function, and  $h(0) = 1$ . By continuity of  $h$  at 0, there is some number  $r > 0$  for which  $h(z)$  is non-zero on the disc  $\{z : |z| < r\}$  centred at 0. It follows that the function

$$g(z) = \frac{3 \cos z}{h(z)}$$

is analytic on  $\{z : |z| < r\}$ . Then

$$f(z) = g(z), \quad \text{for } 0 < |z| < r,$$

so  $f$  has a removable singularity at 0.

(c) The function

$$f(z) = \frac{z^2 + 3iz - 2}{z^2 + 4} = \frac{(z+2i)(z+i)}{(z+2i)(z-2i)}$$

has singularities at  $-2i$  and  $2i$ , and is analytic on  $\mathbb{C} - \{-2i, 2i\}$ .

Observe that

$$f(z) = \frac{z+i}{z-2i}, \quad \text{for } z \in \mathbb{C} - \{-2i, 2i\}.$$

Let  $g$  be the function

$$g(z) = \frac{z+i}{z-2i} \quad (z \in \mathbb{C} - \{2i\}),$$

which is analytic on the disc

$D = \{z : |z - (-2i)| < 4\}$ . Then

$$f(z) = g(z), \quad \text{for } 0 < |z + 2i| < 4,$$

so  $f$  has a removable singularity at  $-2i$ .

The other singularity of  $f$ , at  $2i$ , is not removable because

$$f(z) = \frac{z+i}{z-2i} \rightarrow \infty \text{ as } z \rightarrow 2i,$$

which can be confirmed by applying the Reciprocal Rule.

## Solution to Exercise 1.6

(a) The function  $f(z) = (z-2)/(z+1)$  has a singularity at  $-1$  and is analytic on  $\mathbb{C} - \{-1\}$ .

Observe that

$$f(z) = \frac{g(z)}{z+1}, \quad \text{for } z \in \mathbb{C} - \{-1\},$$

where  $g(z) = z-2$ , an entire function. Since  $g(-1) \neq 0$ ,  $f$  has a simple pole at  $-1$ .

(b) The function  $f(z) = (\cos z)/z$  has a singularity at  $0$  and is analytic on  $\mathbb{C} - \{0\}$ .

Now

$$f(z) = \frac{g(z)}{z}, \quad \text{for } z \in \mathbb{C} - \{0\},$$

where  $g$  is the cosine function.

Since  $g$  is entire and  $g(0) \neq 0$ ,  $f$  has a simple pole at  $0$ .

(c) The function  $f(z) = z/\sin z$  has singularities at  $0, \pm\pi, \pm2\pi, \pm3\pi, \dots$  and is analytic on  $\mathbb{C} - \{k\pi : k \in \mathbb{Z}\}$ .

The singularity at  $0$  is not a simple pole because

$$\lim_{z \rightarrow 0} f(z) = \lim_{z \rightarrow 0} \left( \frac{\sin z}{z} \right)^{-1} = 1.$$

(It is a removable singularity.)

Suppose now that  $k$  is a non-zero integer. Observe that  $f$  is analytic on each punctured disc  $D_k = \{z : 0 < |z - k\pi| < \pi\}$ .

Using the hint, we have

$$\begin{aligned} f(z) &= \frac{(-1)^k z}{\sin(z - k\pi)} \\ &= \frac{(-1)^k z}{(z - k\pi) - \frac{(z - k\pi)^3}{3!} + \frac{(z - k\pi)^5}{5!} - \dots}, \end{aligned}$$

for  $0 < |z - k\pi| < \pi$ . Thus

$$f(z) = \frac{g_k(z)}{z - k\pi}, \quad \text{for } 0 < |z - k\pi| < \pi,$$

where  $g_k$  is the function

$$g_k(z) = \frac{(-1)^k z}{1 - \frac{(z - k\pi)^2}{3!} + \frac{(z - k\pi)^4}{5!} - \dots}.$$

Since  $g_k$  is analytic at  $k\pi$  and  $g_k(k\pi) \neq 0$ ,  $f$  has a simple pole at  $k\pi$ , for each  $k \in \mathbb{Z} - \{0\}$ .

*Remark:* Remember that if a function  $g$  is analytic at  $\alpha$ , then  $g$  is analytic on some open disc  $\{z : |z - \alpha| < r\}$ .

## Solution to Exercise 1.7

(a) The function

$$f(z) = \frac{z+2}{z^4(z^2-4)^3} = \frac{z+2}{z^4(z-2)^3(z+2)^3}$$

has singularities at  $0, 2$  and  $-2$ . Also,  $f$  is analytic on the punctured discs

$$\begin{aligned} A &= \{z : 0 < |z| < 2\}, \\ B &= \{z : 0 < |z - 2| < 2\}, \\ C &= \{z : 0 < |z + 2| < 2\}. \end{aligned}$$

We deal with the singularities one at a time.

First,

$$f(z) = \frac{g_A(z)}{z^4}, \quad \text{for } 0 < |z| < 2,$$

where  $g_A$  is the function

$$g_A(z) = \frac{z+2}{(z^2-4)^3}.$$

Since  $g_A$  is analytic on  $\{z : |z| < 2\}$  and  $g(0) \neq 0$ ,  $f$  has a pole of order 4 at  $0$ .

Second,

$$f(z) = \frac{g_B(z)}{(z-2)^3}, \quad \text{for } 0 < |z - 2| < 2,$$

where  $g_B$  is the function

$$g_B(z) = \frac{1}{z^4(z+2)^2}.$$

Since  $g_B$  is analytic on  $\{z : |z - 2| < 2\}$  and  $g_B(2) \neq 0$ ,  $f$  has a pole of order 3 at 2.

Third,

$$f(z) = \frac{g_C(z)}{(z+2)^2}, \quad \text{for } 0 < |z+2| < 2,$$

where  $g_C$  is the function

$$g_C(z) = \frac{1}{z^4(z-2)^3}.$$

Since  $g_C$  is analytic on  $\{z : |z+2| < 2\}$  and  $g_C(-2) \neq 0$ ,  $f$  has a pole of order 2 at  $-2$ .

(b) The function  $f(z) = z/\sin^3 z$  has singularities at  $0, \pm\pi, \pm2\pi, \dots$ . Also,  $f$  is analytic on each punctured disc

$$D_k = \{z : 0 < |z - k\pi| < \pi\},$$

where  $k \in \mathbb{Z}$ .

For the singularity at 0, we have

$$\begin{aligned} f(z) &= \frac{z}{\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots\right)^3} \\ &= \frac{z}{z^3 \left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots\right)^3} \\ &= \frac{g_0(z)}{z^2}, \end{aligned}$$

for  $0 < |z| < \pi$ , where  $g_0$  is the function

$$g_0(z) = \frac{1}{\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots\right)^3}.$$

Since  $g_0$  is analytic at 0 and  $g_0(0) \neq 0$ ,  $f$  has a pole of order 2 at 0.

Next we consider the singularity at  $k\pi$ , where  $k \in \mathbb{Z} - \{0\}$ .

Since  $\sin(z - k\pi) = (-1)^k \sin z$ , we see that

$$\begin{aligned} f(z) &= \frac{(-1)^{3k} z}{\sin^3(z - k\pi)} \\ &= \frac{(-1)^{3k} z}{\left((z - k\pi) - \frac{(z - k\pi)^3}{3!} + \dots\right)^3} \\ &= \frac{g_k(z)}{(z - k\pi)^3}, \end{aligned}$$

for  $0 < |z - k\pi| < \pi$ , where

$$g_k(z) = \frac{(-1)^{3k} z}{\left(1 - \frac{(z - k\pi)^2}{3!} + \frac{(z - k\pi)^4}{5!} - \dots\right)^3}.$$

Since  $g_k$  is analytic at  $k\pi$  and  $g_k(k\pi) \neq 0$ ,  $f$  has a pole of order 3 at  $k\pi$ , for each  $k \in \mathbb{Z} - \{0\}$ .

## Solution to Exercise 1.8

(a) The function

$$f(z) = \frac{1}{(z-1)(z-3)^2}$$

has a simple pole at 1, has a pole of order 2 at 3, and is analytic elsewhere. It has no essential singularities.

(b) The function  $f(z) = e^{1/z}$  is analytic everywhere except at the point 0, which is a singularity of  $f$ . In fact, 0 is an essential singularity of  $f$ .

To see this, consider the sequence

$$z_n = \frac{1}{n\pi i}, \quad n = 1, 2, \dots$$

Now,  $z_n \rightarrow 0$  as  $n \rightarrow \infty$ , and

$$e^{1/z_n} = e^{n\pi i} = (-1)^n.$$

Therefore the sequence  $(f(z_n))$  does not tend to a finite limit or to  $\infty$ , so  $f(z)$  does not tend to a finite limit or to  $\infty$  as  $z \rightarrow 0$ . Hence, by Theorem 1.2,  $f$  has an essential singularity at 0.

(c) The function

$$f(z) = \frac{z+i}{z(z^2+2iz-1)} = \frac{z+i}{z(z+i)^2} = \frac{1}{z(z+i)}$$

has simple poles at 0 and  $-i$ , and is analytic elsewhere. It has no essential singularities.

## Solution to Exercise 1.9

(a) The function  $f$  has singularities at 0 and  $i$ . Also,  $f$  is analytic on  $D_1 = \{z : 0 < |z| < 1\}$  and  $D_2 = \{z : 0 < |z - i| < 1\}$ .

Now

$$f(z) = \frac{g_1(z)}{z^5}, \quad \text{for } z \in D_1,$$

where  $g_1$  is the function

$$g_1(z) = \frac{1}{(z-i)^2}.$$

Since  $g_1$  is analytic on  $\{z : |z| < 1\}$  and  $g_1(0) \neq 0$ ,  $f$  has a pole of order 5 at 0.

Also,

$$f(z) = \frac{g_2(z)}{(z-i)^2}, \quad \text{for } z \in D_2,$$

where  $g_2$  is the function

$$g_2(z) = \frac{1}{z^5}.$$

Since  $g_2$  is analytic on  $\{z : |z-i| < 1\}$  and  $g_2(i) \neq 0$ ,  $f$  has a pole of order 2 at  $i$ .

**(b)** The function  $f$  has singularities at 0 and  $i$ . Also,  $f$  is analytic on  $D_1 = \{z : 0 < |z| < 1\}$  and  $D_2 = \{z : 0 < |z-i| < 1\}$ .

Now

$$f(z) = \frac{(z+i)(z-i)}{z(z-i)} = \frac{z+i}{z}, \quad \text{for } z \in D_1,$$

so

$$f(z) = \frac{g_1(z)}{z}, \quad \text{for } z \in D_1,$$

where  $g_1$  is the function

$$g_1(z) = z + i.$$

Since  $g_1$  is entire and  $g_1(0) \neq 0$ ,  $f$  has a simple pole at 0.

Also,

$$f(z) = g_2(z), \quad \text{for } z \in D_2,$$

where  $g_2$  is the function

$$g_2(z) = \frac{z+i}{z}.$$

Since  $g_2$  is analytic on  $\{z : |z-i| < 1\}$ ,  $f$  has a removable singularity at  $i$ .

**(c)** The function  $f$  has a singularity at 0 and is analytic on  $\mathbb{C} - \{0\}$ .

Now, using the Taylor series about 0 for  $\sinh$  (one of the basic Taylor series listed in Subsection 3.2 of Unit B3), we have

$$f(z) = \frac{z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots}{z^4},$$

so

$$f(z) = \frac{g(z)}{z^3}, \quad \text{for } z \in \mathbb{C} - \{0\},$$

where  $g$  is the function

$$g(z) = 1 + \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots.$$

Since  $g$  is entire and  $g(0) \neq 0$ ,  $f$  has a pole of order 3 at 0.

**(d)** The function  $f$  has a singularity at 0 and is analytic on  $\mathbb{C} - \{0\}$ . The singularity at 0 is an essential singularity.

To see this, observe that  $\sinh(1/z) = -i \sin(i/z)$ , and consider the sequence

$$z_n = \frac{2i}{(2n+1)\pi}, \quad n = 1, 2, \dots$$

Then  $z_n \rightarrow 0$  as  $n \rightarrow \infty$ , and

$$\begin{aligned} \sinh(1/z_n) &= -i \sin(i/z_n) \\ &= -i \sin((n+1/2)\pi) \\ &= -i(-1)^n. \end{aligned}$$

Therefore the sequence  $(f(z_n))$  does not tend to a limit or to  $\infty$ , so  $f(z)$  does not tend to a limit or to  $\infty$  as  $z \rightarrow 0$ . Hence, by Theorem 1.2,  $f$  has an essential singularity at 0.

**(e)** The function  $f$  has a singularity at 0 and is analytic on  $\mathbb{C} - \{0\}$ .

Now

$$\begin{aligned} f(z) &= \frac{\left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right) - 1}{z} \\ &= \frac{z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots}{z}, \end{aligned}$$

so

$$f(z) = g(z), \quad \text{for } z \in \mathbb{C} - \{0\},$$

where  $g$  is the function

$$g(z) = 1 + \frac{z}{2!} + \frac{z^2}{3!} + \cdots.$$

Since  $g$  is entire,  $f$  has a removable singularity at 0.

**(f)** The function  $f(z) = e^{z-1}$  has no singularities, since it is entire.

**(g)** The function

$$f(z) = \cot z = \frac{\cos z}{\sin z}$$

has singularities at  $0, \pm\pi, \pm2\pi, \pm3\pi, \dots$

Also,  $f$  is analytic on each punctured disc

$$D_k = \{z : 0 < |z - k\pi| < \pi\},$$

where  $k \in \mathbb{Z}$ .

Let  $k \in \mathbb{Z}$ . Since  $\sin(z - k\pi) = (-1)^k \sin z$ , we have

$$\begin{aligned} f(z) &= \frac{(-1)^k \cos z}{\sin(z - k\pi)} \\ &= \frac{(-1)^k \cos z}{(z - k\pi) - \frac{(z - k\pi)^3}{3!} + \frac{(z - k\pi)^5}{5!} - \dots}, \end{aligned}$$

for  $0 < |z - k\pi| < \pi$ . Thus

$$f(z) = \frac{g_k(z)}{z - k\pi}, \quad \text{for } 0 < |z - k\pi| < \pi,$$

where  $g_k$  is the function

$$g_k(z) = \frac{(-1)^k \cos z}{1 - \frac{(z - k\pi)^2}{3!} + \frac{(z - k\pi)^4}{5!} - \dots}.$$

Since  $g_k$  is analytic at  $k\pi$  and  $g_k(k\pi) \neq 0$ ,  $f$  has a simple pole at  $k\pi$ , for each  $k \in \mathbb{Z}$ .

(h) The function  $f(z) = 1/(e^z - 1)$  has singularities at  $0, \pm 2\pi i, \pm 4\pi i, \dots$ . Also,  $f$  is analytic on each punctured disc

$$D_k = \{z : 0 < |z - 2k\pi i| < 2\pi\},$$

where  $k \in \mathbb{Z}$ .

Let  $k \in \mathbb{Z}$ . Using the hint, we have

$$\begin{aligned} f(z) &= \frac{1}{e^{z-2k\pi i} - 1} \\ &= \frac{1}{(z - 2k\pi i) + \frac{(z - 2k\pi i)^2}{2!} + \dots}, \end{aligned}$$

for  $0 < |z - 2k\pi i| < 2\pi$ . Thus

$$f(z) = \frac{g_k(z)}{z - 2k\pi i}, \quad \text{for } 0 < |z - 2k\pi i| < 2\pi,$$

where  $g_k$  is the function

$$g_k(z) = \frac{1}{1 + \frac{(z - 2k\pi i)}{2!} + \frac{(z - 2k\pi i)^2}{3!} + \dots}.$$

Since  $g_k$  is analytic at  $2k\pi i$  and  $g_k(2k\pi i) \neq 0$ ,  $f$  has a simple pole at  $2k\pi i$ , for each  $k \in \mathbb{Z}$ .

## Solution to Exercise 2.1

(a) Observe that  $f(z)$  is equal to

$$\begin{aligned} &\frac{1}{z^4} \left( 1 + 2z + \frac{(2z)^2}{2!} + \frac{(2z)^3}{3!} + \frac{(2z)^4}{4!} + \frac{(2z)^5}{5!} + \frac{(2z)^6}{6!} + \dots \right) \\ &= \frac{1}{z^4} + \frac{2}{z^3} + \frac{2^2}{2!z^2} + \frac{2^3}{3!z} + \frac{2^4}{4!} + \frac{2^5}{5!}z + \frac{2^6}{6!}z^2 + \dots. \end{aligned}$$

Hence the analytic part is

$$\frac{2^4}{4!} + \frac{2^5}{5!}z + \frac{2^6}{6!}z^2 + \dots$$

and the singular part is

$$\frac{1}{z^4} + \frac{2}{z^3} + \frac{2^2}{2!z^2} + \frac{2^3}{3!z}.$$

(b) We have

$$\begin{aligned} f(z) &= \left( 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots \right) \\ &\quad - \left( 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \dots \right) \\ &= \dots - \frac{1}{3!z^3} - \frac{1}{2!z^2} - \frac{1}{z} + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots. \end{aligned}$$

Hence the analytic part is

$$z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$$

and the singular part is

$$-\frac{1}{z} - \frac{1}{2!z^2} - \frac{1}{3!z^3} - \dots.$$

(c) We have

$$\begin{aligned} f(z) &= \frac{1}{z} \left( z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots \right) \\ &= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots. \end{aligned}$$

Hence the analytic part is

$$1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots$$

and the singular part is 0.

## Solution to Exercise 2.2

(a) The analytic part of this series is 1; it is defined for all  $z \in \mathbb{C}$ .

The singular part is

$$\frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \dots.$$

Now,

$$\left| \frac{1/n!}{1/(n+1)!} \right| = n+1 \rightarrow \infty \text{ as } n \rightarrow \infty.$$

Hence, by the Radius of Convergence Formula (Theorem 2.2 of Unit B3), the singular part converges for all  $z \neq 0$ .

Thus the annulus of convergence of the given extended power series is  $\mathbb{C} - \{0\}$ .

(b) The analytic part of this series is

$$1 + z + z^2 + z^3 + \cdots,$$

which converges for  $|z| < 1$ .

The singular part is

$$\frac{1}{z^2} + \frac{1}{z},$$

which is defined for all  $z \neq 0$ .

Thus the annulus of convergence of the given extended power series is  $\{z : 0 < |z| < 1\}$ .

### Solution to Exercise 2.3

(a) Since

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots, \quad \text{for } z \in \mathbb{C},$$

we have

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots,$$

for  $z \in \mathbb{C} - \{0\}$ .

(b) Using the list of basic Taylor series from Subsection 3.2 of Unit B3, we see that

$$\sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots, \quad \text{for } z \in \mathbb{C}.$$

Hence

$$\begin{aligned} \frac{\sinh 2z}{z^2} &= \frac{1}{z^2} \left( 2z + \frac{(2z)^3}{3!} + \frac{(2z)^5}{5!} + \cdots \right) \\ &= \frac{2}{z} + \frac{2^3}{3!}z + \frac{2^5}{5!}z^3 + \cdots, \end{aligned}$$

for  $z \in \mathbb{C} - \{0\}$ .

### Solution to Exercise 2.4

Since

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots, \quad \text{for } z \in \mathbb{C},$$

we have

$$z \sin(1/z^2) = \frac{1}{z} - \frac{1}{3!z^5} + \frac{1}{5!z^9} + \cdots,$$

for  $z \in \mathbb{C} - \{0\}$ . This extended power series has infinitely many non-zero coefficients in its singular part. Hence, by Theorem 2.2(c),  $f$  has an essential singularity at 0.

### Solution to Exercise 2.5

(a) Since the singularities of  $f(z) = 1/(z(z-1))$  are at 0 and 1,  $f$  has two Laurent series about 0. The annuli of convergence are

$$\{z : 0 < |z| < 1\} \quad \text{and} \quad \{z : |z| > 1\}.$$

(b) We write

$$f(z) = -\frac{1}{z} \times \frac{1}{1-z}$$

and note that

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots, \quad \text{for } |z| < 1,$$

and

$$\begin{aligned} \frac{1}{1-z} &= -\frac{1}{z} \times \frac{1}{1-1/z} \\ &= -\frac{1}{z} \left( 1 + \frac{1}{z} + \frac{1}{z^2} + \cdots \right), \quad \text{for } |z| > 1. \end{aligned}$$

For  $0 < |z| < 1$ , we have

$$\begin{aligned} f(z) &= -\frac{1}{z} (1 + z + z^2 + z^3 + \cdots) \\ &= -\frac{1}{z} - 1 - z - z^2 - \cdots. \end{aligned}$$

For  $|z| > 1$ , we have

$$\begin{aligned} f(z) &= -\frac{1}{z} \left( -\frac{1}{z} \left( 1 + \frac{1}{z} + \frac{1}{z^3} + \cdots \right) \right) \\ &= \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \cdots. \end{aligned}$$

### Solution to Exercise 2.6

We first express  $f(z)$  in partial fractions:

$$f(z) = \frac{1}{z-1} - \frac{1}{z+3}.$$

The function  $g(z) = 1/(z-1)$  is analytic on

$$G_1 = \{z : |z| < 1\} \quad \text{and} \quad G_2 = \{z : |z| > 1\}.$$

The function  $h(z) = 1/(z+3)$  is analytic on

$$H_1 = \{z : |z| < 3\} \quad \text{and} \quad H_2 = \{z : |z| > 3\}.$$

(a) Now  $A = \{z : |z| < 1\}$  is such that

$$A \subseteq G_1 \quad \text{and} \quad A \subseteq H_1.$$

The Laurent series for  $g$  on  $G_1$  is

$$\begin{aligned} g(z) &= -\frac{1}{1-z} \\ &= -1 - z - z^2 - z^3 - \dots, \quad \text{for } |z| < 1. \end{aligned}$$

The Laurent series for  $h$  on  $H_1$  is

$$\begin{aligned} h(z) &= \frac{1}{3} \times \frac{1}{1+z/3} \\ &= \frac{1}{3} \left( 1 - \frac{z}{3} + \frac{z^2}{9} - \frac{z^3}{27} + \dots \right) \\ &= \frac{1}{3} - \frac{z}{9} + \frac{z^2}{27} - \frac{z^3}{81} + \dots, \quad \text{for } |z| < 3. \end{aligned}$$

Since  $f(z) = g(z) - h(z)$ , we obtain

$$f(z) = -\frac{4}{3} - \frac{8}{9}z - \frac{28}{27}z^2 - \frac{80}{81}z^3 - \dots,$$

for  $|z| < 1$ .

(b) Now  $B = \{z : 1 < |z| < 3\}$  is such that

$$B \subseteq G_2 \quad \text{and} \quad B \subseteq H_1.$$

The Laurent series for  $g$  on  $G_2$  is

$$\begin{aligned} g(z) &= \frac{1}{z} \times \frac{1}{1-1/z} \\ &= \frac{1}{z} \left( 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \dots \right) \\ &= \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \dots, \quad \text{for } |z| > 1. \end{aligned}$$

The Laurent series for  $h$  on  $H_1$  is, from part (a),

$$h(z) = \frac{1}{3} - \frac{z}{9} + \frac{z^2}{27} - \frac{z^3}{81} + \dots, \quad \text{for } |z| < 3.$$

Since  $f(z) = g(z) - h(z)$ , we obtain

$$\begin{aligned} f(z) &= \dots + \frac{1}{z^4} + \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} \\ &\quad - \frac{1}{3} + \frac{z}{9} - \frac{z^2}{27} + \frac{z^3}{81} - \dots, \end{aligned}$$

for  $1 < |z| < 3$ .

(c) Now  $C = \{z : |z| > 3\}$  is such that

$$C \subseteq G_2 \quad \text{and} \quad C \subseteq H_2.$$

The Laurent series for  $g$  on  $G_2$  is, from part (b),

$$g(z) = \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \dots, \quad \text{for } |z| > 1,$$

and the Laurent series for  $h$  on  $H_2$  is

$$\begin{aligned} h(z) &= \frac{1}{z} \times \frac{1}{1+3/z} \\ &= \frac{1}{z} \left( 1 - \frac{3}{z} + \frac{9}{z^2} - \frac{27}{z^3} + \dots \right) \\ &= \frac{1}{z} - \frac{3}{z^2} + \frac{9}{z^3} - \frac{27}{z^4} + \dots, \quad \text{for } |z| > 3. \end{aligned}$$

Since  $f(z) = g(z) - h(z)$ , we obtain

$$f(z) = \frac{4}{z^2} - \frac{8}{z^3} + \frac{28}{z^4} - \dots, \quad \text{for } |z| > 3.$$

## Solution to Exercise 2.7

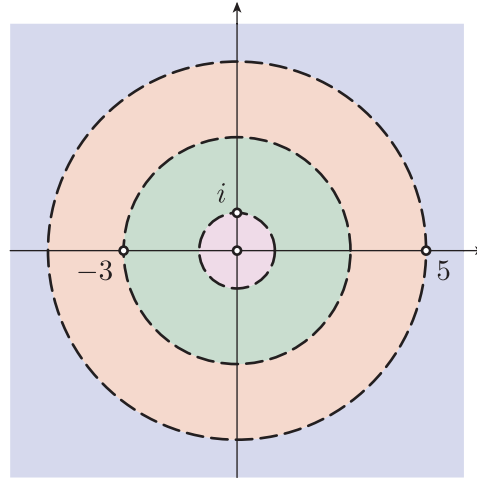
The singularities of the function

$$f(z) = \frac{z-3i}{z(z-i)(z+3)(z-5)}$$

are at  $0, i, -3$  and  $5$ . Hence  $f$  has four Laurent series about  $0$ . The annuli of convergence are

$$\begin{aligned} \{z : 0 < |z| < 1\}, \quad \{z : 1 < |z| < 3\}, \\ \{z : 3 < |z| < 5\}, \quad \{z : |z| > 5\}, \end{aligned}$$

as shown in the figure.



## Solution to Exercise 2.8

(a) Let  $w = z - 2$ . Then, for  $z \neq 2$  (equivalently,  $w \neq 0$ ),

$$\begin{aligned} f(z) &= \frac{\cos w}{w^2} \\ &= \frac{1}{w^2} \left( 1 - \frac{w^2}{2!} + \frac{w^4}{4!} - \dots \right) \\ &= \frac{1}{w^2} - \frac{1}{2!} + \frac{w^2}{4!} - \dots \\ &= \frac{1}{(z-2)^2} - \frac{1}{2!} + \frac{(z-2)^2}{4!} - \dots, \end{aligned}$$

for  $z \in \mathbb{C} - \{2\}$ .

(b) Let  $w = z - 2$ , so  $z = w + 2$ . Then, for  $z \neq 2$  (equivalently,  $w \neq 0$ ),

$$\begin{aligned} f(z) &= (w + 2) \cos\left(\frac{1}{w}\right) \\ &= (w + 2) \left(1 - \frac{1}{2!} \left(\frac{1}{w}\right)^2 + \frac{1}{4!} \left(\frac{1}{w}\right)^4 - \dots\right) \\ &= \left(w - \frac{1}{2!w} + \frac{1}{4!w^3} - \dots\right) \\ &\quad + \left(2 - \frac{2}{2!w^2} + \frac{2}{4!w^4} - \dots\right) \\ &= w + 2 - \frac{1}{2!w} - \frac{2}{2!w^2} + \frac{1}{4!w^3} + \frac{2}{4!w^4} - \dots, \\ &= (z - 2) + 2 - \frac{1}{2(z - 2)} - \frac{1}{(z - 2)^2} \\ &\quad + \frac{1}{24(z - 2)^3} + \frac{1}{12(z - 2)^4} - \dots, \end{aligned}$$

for  $z \in \mathbb{C} - \{2\}$ .

(c) Let  $w = z - 2$ , so  $z = w + 2$ . Then, for  $z \neq 2, -2$  (equivalently,  $w \neq 0, -4$ ),

$$\begin{aligned} f(z) &= \frac{1}{(z - 2)(z + 2)} \\ &= \frac{1}{w(w + 4)} \\ &= \frac{1}{4w} \times \frac{1}{1 + w/4} \\ &= \frac{1}{4w} \left(1 - \left(\frac{w}{4}\right) + \left(\frac{w}{4}\right)^2 - \left(\frac{w}{4}\right)^3 + \dots\right) \\ &= \frac{1}{4w} - \frac{1}{16} + \frac{w}{64} - \frac{w^2}{256} + \dots, \end{aligned}$$

for  $0 < |w| < 4$ . Therefore

$$f(z) = \frac{1}{4(z - 2)} - \frac{1}{16} + \frac{z - 2}{64} - \frac{(z - 2)^2}{256} + \dots,$$

for  $0 < |z - 2| < 4$ .

## Solution to Exercise 2.9

The function

$$f(z) = \frac{1}{z(z + 3)(z + 6)}$$

has singularities at 0, -3 and -6.

(a)  $f$  has two Laurent series about -3; the annuli of convergence are

$$A = \{z : 0 < |z + 3| < 3\},$$

$$B = \{z : |z + 3| > 3\}.$$

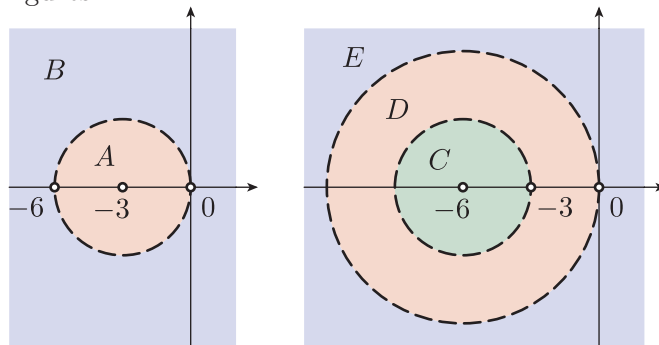
(b)  $f$  has three Laurent series about -6; the annuli of convergence are

$$C = \{z : 0 < |z + 6| < 3\},$$

$$D = \{z : 3 < |z + 6| < 6\},$$

$$E = \{z : |z + 6| > 6\}.$$

The two sets of annuli of convergence from parts (a) and (b) are shown in the following figures.



## Solution to Exercise 2.10

(a) Let  $w = z + 3$ , so  $z = w - 3$ . Then, for  $z \neq 1, -3$ , we have

$$\begin{aligned} f(z) &= \frac{4}{(w - 4)w} \\ &= -\frac{1}{w} \times \frac{1}{1 - w/4} \\ &= -\frac{1}{w} \left(1 + \frac{w}{4} + \left(\frac{w}{4}\right)^2 + \left(\frac{w}{4}\right)^3 + \dots\right) \\ &= -\frac{1}{w} - \frac{1}{4} - \frac{w}{16} - \frac{w^2}{64} - \dots, \end{aligned}$$

for  $0 < |w| < 4$ . Therefore

$$f(z) = -\frac{1}{z + 3} - \frac{1}{4} - \frac{z + 3}{16} - \frac{(z + 3)^2}{64} - \dots,$$

for  $0 < |z + 3| < 4$ .

(b) Let  $w = z + 1$ , so  $z = w - 1$ . Then, for  $z \neq 1, -3$ , we have

$$\begin{aligned} f(z) &= \frac{4}{(w - 2)(w + 2)} \\ &= \frac{4}{w^2 - 4} \\ &= \frac{4}{w^2} \times \frac{1}{1 - 4/w^2} \\ &= \frac{4}{w^2} \left(1 + \frac{4}{w^2} + \left(\frac{4}{w^2}\right)^2 + \dots\right) \\ &= \frac{4}{w^2} + \frac{4^2}{w^4} + \frac{4^3}{w^6} + \dots, \quad \text{for } |w| > 2. \end{aligned}$$

Therefore

$$\begin{aligned} f(z) &= \frac{4}{(z+1)^2} + \frac{4^2}{(z+1)^4} + \frac{4^3}{(z+1)^6} + \cdots \\ &= \frac{4}{(z+1)^2} + \frac{16}{(z+1)^4} + \frac{64}{(z+1)^6} + \cdots, \end{aligned}$$

for  $|z+1| > 2$ .

### Solution to Exercise 2.11

(a) The analytic part of this series is

$$1 - z + z^2 - z^3 + \cdots,$$

which converges for  $|z| < 1$ .

The singular part  $i/z$  is defined for all  $z \neq 0$ .

Thus the annulus of convergence of the given extended power series is  $\{z : 0 < |z| < 1\}$ .

(b) The analytic part of this series is 1, which is defined for all  $z$ .

The singular part is

$$\frac{1}{z} - \frac{1}{z^2} + \frac{1}{z^3} - \cdots,$$

which converges for  $|1/z| < 1$ , that is, for  $|z| > 1$ .

Thus the annulus of convergence of the given extended power series is  $\{z : |z| > 1\}$ .

### Solution to Exercise 2.12

The Laurent series about 0 for the function

$$f(z) = \left(\frac{1}{z} - \frac{1}{z^2}\right) \sin z$$

is given by

$$\begin{aligned} &\left(\frac{1}{z} - \frac{1}{z^2}\right) \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right) \\ &= \left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots\right) - \left(\frac{1}{z} - \frac{z}{3!} + \frac{z^3}{5!} - \cdots\right) \\ &= -\frac{1}{z} + 1 + \frac{z}{3!} - \frac{z^2}{3!} - \frac{z^3}{5!} + \frac{z^4}{5!} + \cdots, \end{aligned}$$

for  $z \neq 0$ . The function  $f$  is represented by this Laurent series on *any* punctured disc with centre 0, for example,  $\{z : 0 < |z| < 1\}$ .

### Solution to Exercise 2.13

(a) The Laurent series about 0 for the function  $f(z) = 1/(z(z-4))$  on  $\{z : 0 < |z| < 4\}$  is given by

$$\begin{aligned} f(z) &= \frac{1}{z(z-4)} \\ &= -\frac{1}{4z} \times \frac{1}{1-z/4} \\ &= -\frac{1}{4z} \left(1 + \frac{z}{4} + \left(\frac{z}{4}\right)^2 + \left(\frac{z}{4}\right)^3 + \cdots\right) \\ &= -\frac{1}{4z} - \frac{1}{16} - \frac{z}{64} - \frac{z^2}{256} - \cdots, \end{aligned}$$

for  $0 < |z| < 4$ .

(b) The Laurent series about 0 for  $f$  on  $\{z : |z| > 4\}$  is given by

$$\begin{aligned} f(z) &= \frac{1}{z(z-4)} \\ &= \frac{1}{z^2} \times \frac{1}{(1-4/z)} \\ &= \frac{1}{z^2} \left(1 + \frac{4}{z} + \left(\frac{4}{z}\right)^2 + \left(\frac{4}{z}\right)^3 + \cdots\right) \\ &= \frac{1}{z^2} + \frac{4}{z^3} + \frac{16}{z^4} + \frac{64}{z^5} + \cdots, \end{aligned}$$

for  $|z| > 4$ .

(c) We obtain the Laurent series about 4 for  $f$  on  $\{z : 0 < |z-4| < 4\}$  by making the substitution  $w = z-4$ . Then  $z = w+4$  and

$$\begin{aligned} f(z) &= \frac{1}{(w+4)w} \\ &= \frac{1}{4w} \times \frac{1}{1+w/4} \\ &= \frac{1}{4w} \left(1 - \frac{w}{4} + \left(\frac{w}{4}\right)^2 - \left(\frac{w}{4}\right)^3 + \cdots\right) \\ &= \frac{1}{4w} - \frac{1}{16} + \frac{w}{64} - \frac{w^2}{256} + \cdots, \end{aligned}$$

for  $0 < |w| < 4$ . Substituting  $w = z-4$  gives

$$f(z) = \frac{1}{4(z-4)} - \frac{1}{16} + \frac{z-4}{64} - \frac{(z-4)^2}{256} + \cdots,$$

for  $0 < |z-4| < 4$ .

## Solution to Exercise 2.14

We first express  $f$  in partial fractions:

$$f(z) = \frac{1}{(z-1)(z-3)} = \frac{-1/2}{z-1} + \frac{1/2}{z-3}.$$

Note that the function  $g(z) = 1/(z-1)$  is analytic on

$$\{z : |z| < 1\} \quad \text{and} \quad \{z : |z| > 1\},$$

and that the function  $h(z) = 1/(z-3)$  is analytic on

$$\{z : |z| < 3\} \quad \text{and} \quad \{z : |z| > 3\}.$$

(a) The Laurent series about 0 for  $f$  on  $\{z : |z| < 1\}$ , which is a subset of  $\{z : |z| < 3\}$ , is

$$\begin{aligned} f(z) &= \frac{1}{2} \times \frac{1}{1-z} - \frac{1}{6} \times \frac{1}{1-z/3} \\ &= \frac{1}{2}(1+z+z^2+\cdots) \\ &\quad - \frac{1}{6}\left(1+\frac{z}{3}+\frac{z^2}{3^2}+\cdots\right) \\ &= \frac{1}{3} + \frac{4}{9}z + \frac{13}{27}z^2 + \cdots, \quad \text{for } |z| < 1. \end{aligned}$$

(b) The Laurent series about 0 for  $f$  on  $\{z : |z| > 3\}$ , which is a subset of  $\{z : |z| > 1\}$ , is

$$\begin{aligned} f(z) &= -\frac{1}{2z} \times \frac{1}{1-1/z} + \frac{1}{2z} \times \frac{1}{1-3/z} \\ &= -\frac{1}{2z} \left(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots\right) \\ &\quad + \frac{1}{2z} \left(1 + \frac{3}{z} + \frac{3^2}{z^2} + \frac{3^3}{z^3} + \cdots\right) \\ &= \frac{1}{z^2} + \frac{4}{z^3} + \frac{13}{z^4} + \cdots, \quad \text{for } |z| > 3. \end{aligned}$$

(c) We obtain the Laurent series about 1 for  $f$  on  $\{z : 0 < |z-1| < 2\}$  by making the substitution  $w = z-1$ . Then  $z = w+1$  and

$$\begin{aligned} f(z) &= \frac{1}{w(w-2)} \\ &= -\frac{1}{2w} \times \frac{1}{1-w/2} \\ &= -\frac{1}{2w} \left(1 + \frac{w}{2} + \frac{w^2}{2^2} + \frac{w^3}{2^3} + \cdots\right) \\ &= -\frac{1}{2w} - \frac{1}{2^2} - \frac{w}{2^3} - \frac{w^2}{2^4} - \cdots, \end{aligned}$$

for  $0 < |w| < 2$ . Therefore

$$f(z) = -\frac{1}{2(z-1)} - \frac{1}{4} - \frac{(z-1)}{8} - \frac{(z-1)^2}{16} - \cdots,$$

for  $0 < |z-1| < 2$ .

## Solution to Exercise 2.15

This solution makes use of Taylor series about 0 for  $\cos$  and  $\sinh$ , which we recorded in the list of basic Taylor series in Subsection 3.2 of Unit B3.

(a) The Laurent series about 0 for the function  $f(z) = \cos(1/z)$  is

$$\cos \frac{1}{z} = 1 - \frac{1}{2!z^2} + \frac{1}{4!z^4} - \cdots, \quad \text{for } |z| > 0,$$

which has infinitely many non-zero coefficients in its singular part. Hence, by Theorem 2.2(c),  $f$  has an essential singularity at 0.

(b) The Laurent series about 0 for the function  $f(z) = z \sinh(1/z)$  is

$$\begin{aligned} z \sinh \frac{1}{z} &= z \left( \frac{1}{z} + \frac{1}{3!} \left( \frac{1}{z} \right)^3 + \frac{1}{5!} \left( \frac{1}{z} \right)^5 + \cdots \right) \\ &= 1 + \frac{1}{3!z^2} + \frac{1}{5!z^4} + \cdots, \quad \text{for } |z| > 0, \end{aligned}$$

which has infinitely many non-zero coefficients in its singular part. Hence, by Theorem 2.2(c),  $f$  has an essential singularity at 0.

## Solution to Exercise 3.1

(a) Here  $f(z) = (\sin^2 z)/z^2$  and  $\alpha = 0$ . We have

$$\begin{aligned} \lim_{z \rightarrow 0} z f(z) &= \lim_{z \rightarrow 0} z \left( \frac{\sin z}{z} \right)^2 \\ &= 0 \times 1^2 = 0. \end{aligned}$$

Hence condition (d) is verified.

(b) Here  $f(z) = 3z/\tan 3z$  and  $\alpha = 0$ . We have

$$\begin{aligned} \lim_{z \rightarrow 0} z f(z) &= \lim_{z \rightarrow 0} z \times \frac{3z}{\tan 3z} \\ &= \lim_{z \rightarrow 0} z \cos 3z \left( \frac{3z}{\sin 3z} \right) \\ &= 0 \times 1 \times 1 = 0. \end{aligned}$$

Hence condition (d) is verified.

(c) Here  $f(z) = (z^2 + 3iz - 2)/(z^2 + 4)$  and  $\alpha = -2i$ . We have

$$\begin{aligned} \lim_{z \rightarrow -2i} (z + 2i) f(z) &= \lim_{z \rightarrow -2i} (z + 2i) \times \frac{z^2 + 3iz - 2}{z^2 + 4} \\ &= \lim_{z \rightarrow -2i} (z + 2i) \times \frac{(z + 2i)(z + i)}{(z + 2i)(z - 2i)} \\ &= \lim_{z \rightarrow -2i} \frac{(z + 2i)(z + i)}{z - 2i} \\ &= \frac{0 \times (-i)}{-4i} = 0. \end{aligned}$$

Hence condition (d) is verified.

### Solution to Exercise 3.2

(a) The function

$$f(z) = \frac{z + 2}{z^4(z^2 - 4)^3}$$

has a pole of order 4 at 0, a pole of order 3 at 2 and a pole of order 2 at  $-2$  (see Exercise 1.7(a)). We have

$$\begin{aligned} \lim_{z \rightarrow 0} z^4 f(z) &= \lim_{z \rightarrow 0} \frac{z + 2}{(z^2 - 4)^3} \\ &= \frac{2}{(-4)^3} \neq 0, \end{aligned}$$

$$\begin{aligned} \lim_{z \rightarrow 2} (z - 2)^3 f(z) &= \lim_{z \rightarrow 2} \frac{(z - 2)^3(z + 2)}{z^4(z^2 - 4)^3} \\ &= \lim_{z \rightarrow 2} \frac{1}{z^4(z + 2)^2} \\ &= \frac{1}{2^4 \times 4^2} \neq 0, \end{aligned}$$

$$\begin{aligned} \lim_{z \rightarrow -2} (z + 2)^2 f(z) &= \lim_{z \rightarrow -2} \frac{(z + 2)^3}{z^4(z^2 - 4)^3} \\ &= \lim_{z \rightarrow -2} \frac{1}{z^4(z - 2)^3} \\ &= \frac{1}{(-2)^4 \times (-4)^3} \neq 0. \end{aligned}$$

Hence, in each case, condition (b) is verified.

(b) The function  $f(z) = z/\sin^3 z$  has a pole of order 2 at 0 and a pole of order 3 at  $k\pi$ , for each  $k \in \mathbb{Z} - \{0\}$  (see Exercise 1.7(b)). We have

$$\begin{aligned} \lim_{z \rightarrow 0} z^2 f(z) &= \lim_{z \rightarrow 0} \frac{z^3}{\sin^3 z} \\ &= \lim_{z \rightarrow 0} \left( \frac{z}{\sin z} \right)^3 \\ &= 1^3 \neq 0, \end{aligned}$$

and, for  $k \in \mathbb{Z} - \{0\}$ ,

$$\begin{aligned} \lim_{z \rightarrow k\pi} (z - k\pi)^3 f(z) &= \lim_{z \rightarrow k\pi} \frac{(z - k\pi)^3 z}{\sin^3 z} \\ &= \lim_{z \rightarrow k\pi} \frac{(-1)^{3k} z (z - k\pi)^3}{\sin^3(z - k\pi)} \\ &= \lim_{z \rightarrow k\pi} (-1)^{3k} z \left( \frac{z - k\pi}{\sin(z - k\pi)} \right)^3 \\ &= (-1)^k k\pi \times \lim_{z \rightarrow k\pi} \left( \frac{z - k\pi}{\sin(z - k\pi)} \right)^3 \\ &= (-1)^k k\pi \times \lim_{w \rightarrow 0} \left( \frac{w}{\sin w} \right)^3 \\ &= (-1)^k k\pi \times 1^3 \neq 0. \end{aligned}$$

Hence, in each case, condition (b) is verified.

### Solution to Exercise 3.3

(a) Simple pole (by definition, or use Theorem 3.2 ((b)  $\implies$  (a))).

(b) Removable singularity (by Theorem 3.1 ((d)  $\implies$  (a))).

(c) Pole of order 2 (by Theorem 3.2 ((c)  $\implies$  (a))).

### Solution to Exercise 3.4

(a) The function  $f + g$  has a pole of order 5 at  $\alpha$ , since

$$\begin{aligned} \lim_{z \rightarrow \alpha} (z - \alpha)^5 (f(z) + g(z)) &= \lim_{z \rightarrow \alpha} (z - \alpha)^5 f(z) + \lim_{z \rightarrow \alpha} (z - \alpha)^5 g(z) \\ &= \lim_{z \rightarrow \alpha} (z - \alpha)^5 f(z) + \lim_{z \rightarrow \alpha} (z - \alpha)^2 \lim_{z \rightarrow \alpha} (z - \alpha)^3 g(z) \\ &= \lim_{z \rightarrow \alpha} (z - \alpha)^5 f(z) + 0, \end{aligned}$$

which is non-zero.

The result follows from Theorem 3.2 ((b)  $\implies$  (a)).

The function  $fg$  has a pole of order 8 at  $\alpha$  since

$$\begin{aligned} \lim_{z \rightarrow \alpha} (z - \alpha)^8 (fg)(z) &= \left( \lim_{z \rightarrow \alpha} (z - \alpha)^5 f(z) \right) \left( \lim_{z \rightarrow \alpha} (z - \alpha)^3 g(z) \right), \end{aligned}$$

which is non-zero.

The result again follows from Theorem 3.2 ((b)  $\implies$  (a)).

(b)  $fg$  has a pole of order 8 at  $\alpha$ , for reasons similar to those above. However,  $f + g$  need not have a pole of order 4 at  $\alpha$ ; indeed,  $f + g$  need not have a pole at all (for example, if  $g = -f$ ). The most that can be said is that  $f + g$  has a removable singularity or a pole whose order does not exceed 4.

### Solution to Exercise 3.5

The function  $f(z) = e^{1/z}$  has an essential singularity at  $\alpha = 0$ , by Exercise 1.8(b). The result then follows immediately from the Casorati–Weierstrass Theorem by taking

$$w = 1000i, \quad \varepsilon = 10^{-6}, \quad \delta = 10^{-3}.$$

### Solution to Exercise 3.6

In order to show that the function  $g$  has an essential singularity at  $\alpha$ , it is sufficient to show that  $g$  has neither a removable singularity nor a pole at  $\alpha$ .

If  $g$  has a removable singularity at  $\alpha$ , then, by Theorem 3.1 ((a)  $\implies$  (c)),  $g$  is bounded on some punctured open disc with centre  $\alpha$ , say:

$$|g(z)| \leq K, \quad \text{for } 0 < |z - \alpha| < r.$$

Hence

$$|f(z)| \leq \sqrt{K}, \quad \text{for } 0 < |z - \alpha| < r,$$

so, by Theorem 3.1 ((c)  $\implies$  (a)),  $f$  has a removable singularity at  $\alpha$ , which is false.

If  $g$  has a pole at  $\alpha$ , then, by the corollary to Theorem 3.2,

$$g(z) \rightarrow \infty \text{ as } z \rightarrow \alpha.$$

Thus, by the assumption given in the question,

$$f(z) \rightarrow \infty \text{ as } z \rightarrow \alpha,$$

and hence, again by the corollary to Theorem 3.2,  $f$  has a pole at  $\alpha$ , which is false.

Since  $g$  has neither a removable singularity nor a pole at  $\alpha$ , it follows that  $g$  has an essential singularity at  $\alpha$ .

### Solution to Exercise 4.1

Let

$$f(z) = \sinh(1/z) = \frac{1}{z} + \frac{1}{3!z^3} + \frac{1}{5!z^5} + \cdots,$$

for  $z \in \mathbb{C} - \{0\}$ . Then, using equation (4.1),

$$\begin{aligned} \text{(a)} \quad \int_C w^{-5} \sinh(1/w) dw &= \int_C \frac{f(w)}{w^5} dw \\ &= 2\pi i a_4 = 0 \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \int_C w^4 \sinh(1/w) dw &= \int_C \frac{f(w)}{w^{-4}} dw \\ &= 2\pi i a_{-5} = \frac{\pi i}{60}. \end{aligned}$$

### Solution to Exercise 4.2

(a) The function  $f(z) = 1/(z+i)^2$  has a pole of order 2 at the point  $-i$ , and the Laurent series about  $-i$  for  $f$  is simply

$$\frac{1}{(z+i)^2} + \frac{0}{z+i} + 0 + \cdots, \quad \text{for } z \in \mathbb{C} - \{-i\},$$

so  $\text{Res}(f, -i) = 0$ . Since  $C$  has centre  $-i$ , we obtain, by equation (4.3),

$$\int_C \frac{1}{(z+i)^2} dz = 0.$$

*Remark:* This result could also have been obtained by using the Closed Contour Theorem.

(b) The function  $f(z) = (\sin 2z)/z^4$  has a pole of order 3 at the point 0, and the Laurent series about 0 for  $f$  is

$$\begin{aligned} \frac{1}{z^4} \left( (2z) - \frac{(2z)^3}{3!} + \frac{(2z)^5}{5!} - \cdots \right) \\ = \frac{2}{z^3} - \frac{4}{3z} + \frac{4z}{15} - \cdots, \quad \text{for } z \in \mathbb{C} - \{0\}, \end{aligned}$$

so  $\text{Res}(f, 0) = -4/3$ . Since  $C$  has centre 0, we obtain, by equation (4.3),

$$\int_C \frac{\sin 2z}{z^4} dz = 2\pi i \text{Res}(f, 0) = -\frac{8}{3}\pi i.$$

### Solution to Exercise 4.3

In some of the following parts there are several ways of evaluating the integral, so do not worry if your method differs from ours.

In each part we use  $I$  to denote the relevant integral, and we do not state obvious facts like ‘ $\mathbb{C}$  is a simply connected region’ and ‘the circle  $C$  is a simple-closed contour in  $\mathbb{C}$ ’, where these are relevant to the solution. Also, we use the notation of named results.

(a) We use Cauchy’s Integral Formula with  $f(z) = e^z$ ,  $\alpha = 0$ ,  $\mathcal{R} = \mathbb{C}$  and  $\Gamma = C$ . Then  $f$  is analytic on  $\mathcal{R}$ , and we get

$$I = 2\pi i f(0) = 2\pi i e^0 = 2\pi i.$$

Alternatively, we note that the Laurent series about 0 for the function  $f(z) = e^z/z$  is

$$\frac{1}{z} + 1 + \frac{z}{2!} + \cdots, \quad \text{for } z \in \mathbb{C} - \{0\},$$

so  $\text{Res}(f, 0) = 1$ . Since  $C$  has centre 0, we obtain, by equation (4.3),

$$I = 2\pi i \times 1 = 2\pi i.$$

(b) We use Cauchy’s Theorem. The function  $f(z) = e^z/z$  is analytic on the simply connected region  $\mathcal{R} = \{z : \text{Re } z > \frac{1}{4}\}$ , which contains  $C$ . So  $I = 0$ .

(c) We use the Closed Contour Theorem with  $f(z) = \sec^2 z$ ,  $F(z) = \tan z$ ,  $\mathcal{R} = \{z : 0 < |z - \pi/2| < \pi/2\}$  and  $\Gamma = C$ . Then  $f$  is continuous and has primitive  $F$  on  $\mathcal{R}$ , and  $\Gamma$  is a closed contour in  $\mathcal{R}$ . Hence  $I = 0$ .

(d) The Laurent series about 0 for the function  $f(z) = (\cosh z)/z^5$  is given by

$$\frac{1}{z^5} \left( 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \cdots \right),$$

for  $z \in \mathbb{C} - \{0\}$ , so  $\text{Res}(f, 0) = 1/4!$ . Since  $C$  has centre 0, we obtain, by equation (4.3),

$$I = 2\pi i \times (1/4!) = \pi i/12.$$

Alternatively, we can use Cauchy’s  $n$ th Derivative Formula with  $n = 4$ ,  $f(z) = \cosh z$ ,  $\alpha = 0$ ,  $\mathcal{R} = \mathbb{C}$  and  $\Gamma = C$ . Then  $f$  is analytic on  $\mathcal{R}$ ,  $f^{(4)}(0) = \cosh 0 = 1$ , and we obtain

$$I = \frac{2\pi i}{4!} \times 1 = \pi i/12.$$

(e) We use Cauchy’s First Derivative Formula with  $f(z) = z + \frac{1}{2} \sin 2z$ ,  $\alpha = \pi/4$ ,  $\mathcal{R} = \mathbb{C}$  and  $\Gamma = C$ . Then  $f$  is analytic on  $\mathcal{R}$ ,  $f'(\pi/4) = 1 + \cos(\pi/2) = 1$ , and we obtain

$$I = 2\pi i \times 1 = 2\pi i.$$

(f) The function  $f(z) = z \operatorname{cosec} z = z/\sin z$  has a removable singularity at 0. If we define  $f(0) = \lim_{z \rightarrow 0} f(z) = 1$ , then  $f$  is analytic on the simply connected region  $\mathcal{R} = \{z : |z| < \pi\}$ , which contains  $C$ . Hence, by Cauchy’s Theorem,  $I = 0$ .

(g) The Laurent series about 0 for the function  $f(z) = \exp(1/z^4)$  is given by

$$1 + 1/z^4 + \frac{(1/z^4)^2}{2!} + \cdots, \quad \text{for } z \in \mathbb{C} - \{0\},$$

so  $\text{Res}(f, 0) = 0$ . Since  $C$  has centre 0, we obtain, by equation (4.3),

$$I = 2\pi i \times 0 = 0.$$

(h) The Laurent series about 0 for the function  $f(z) = e^{1/z} \sin(1/z)$  is

$$\left(1 + \frac{1}{z} + \cdots\right) \left(\frac{1}{z} - \frac{1}{3!z^3} + \cdots\right) = \frac{1}{z} + \cdots,$$

so  $\text{Res}(f, 0) = 1$ . Since  $C$  has centre 0, we obtain, by equation (4.3),

$$I = 2\pi i \times 1 = 2\pi i.$$

(i) We use Cauchy’s First Derivative Formula with  $f(z) = e^z/(z-1)$ ,  $\alpha = 0$ ,  $\mathcal{R} = \{z : \text{Re } z < \frac{3}{4}\}$  and  $\Gamma = C$ . Then  $f$  is analytic on the simply connected region  $\mathcal{R}$ , which contains  $C$ , and  $f'(0) = -2$ . We obtain

$$I = 2\pi i \times (-2) = -4\pi i.$$

(j) Since

$$\frac{1}{z^2 - 1} = \frac{1/2}{z - 1} - \frac{1/2}{z + 1},$$

we have

$$I = \int_C \frac{\sin \pi z}{z - 1} dz - \int_C \frac{\sin \pi z}{z + 1} dz.$$

We use Cauchy’s Integral Formula with  $f(z) = \sin \pi z$ ,  $\alpha = 1$  and  $-1$ ,  $\mathcal{R} = \mathbb{C}$  and  $\Gamma = C$ , to evaluate the two integrals. Then  $f$  is analytic on  $\mathcal{R}$ , and we obtain

$$I = 2\pi i \sin(\pi) - 2\pi i \sin(-\pi) = 0.$$

## References

Bottazzini, U. and Gray, J. (2013) *Hidden Harmony – Geometric Fantasies: The Rise of Complex Function Theory*, Sources and Studies in the History of Mathematics and Physical Sciences, New York, Springer.

Katz, V. J. (1993) *A History of Mathematics*, New York, HarperCollins College Publishers.

Veblen, O. (1905) ‘Theory on plane curves in non-metrical analysis situs’, *Transactions of the American Mathematical Society*, vol. 6, no. 1, pp. 83–98.

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